Worthiness Based Social Scaling

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Abstract
The construction of a set of scales is delineated, for evaluating the performance of social agents (e.g. providers of services as hospitals, schools, etc.) conditionally on "reference states" $x := X \in \{x_1, \ldots, x_R\}$ of the governed individuals. Each scale is associated to an index which uses conditional "worthiness increases" $\omega_{l|x}$, between the levels of an ordinal outcome indicator $Y := l \in (0, 1, .., L)$. This indicator was been defined on a scheduled, by the policy-maker (PM), chain of hierarchically ordered goals. The "worthiness increases" are interpreted by modeling interrelated latent evolutionary processes, on the scheduled goal chain, up to hyperparameters γ which are driven by conditions x. Then, to standardize the set of scales on a given "reference behavior", a pseudo-Bayesian (see [1]) method is used which elicits value γ^* by minimizing "residual from updating" (see [4]). It norms the model specifications on the "reference data" of the (chosen a priori) "standard agent". Finally, adhering to general requirements in rational choices from the decision theory, a standardized worthiness-based index can be implemented, which takes into input the agents actual data.

Key words: performance index, ordinal scaling, worthiness

1 Indexing worthiness

The performance of any social agent u is associated to the "social behavior", described by the set of distributions (e.g. see table 1) $p_{|x}[u] := (p_{0|x}, p_{1|x}, \cdots, p_{L|x})[u]$, which were realized on the set of the individuals that u governs, upon the levels of

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an ordinal classifier of outcome Y varying the status $x := X \in \{x_1, \dots, x_R\}$ of the governed individuals.

agent A1	performance level (Y)					**agent A2**	performance level(Y)					**agent A3**	performance level(Y)					**agent A4**	performance level (Y)				
status (X)	I	II	III	IV	V	status (X)	I	II	III	IV	V	status (X)	I	II	III	IV	V	status (X)	I	II	III	IV	V
x1	0	0	0	0	0	x1	2	2	0	0	0	x1	0	3	1	0	0	x1	1	3	0	0	0
x2	2	9	5	0	0	x2	3	13	9	1	0	x2	2	24	16	0	0	x2	16	37	18	0	0
x3	0	3	18	0	0	x3	1	20	31	4	1	x3	0	20	48	1	0	x3	0	36	59	4	1
x4	0	3	17	3	0	x4	1	3	59	8	1	x4	0	2	53	3	0	x4	0	12	107	10	0
x5	0	0	14	9	4	x5	0	6	48	18	3	x5	0	0	49	30	2	x5	0	0	87	43	4

Table 1 Example. Actual data of the social agents to be evaluated

reference agent A0	performance level (Y)				
status (X)	I	II	III	V	V
x1	3	8	1	0	0
x2	23	83	48	1	0
x3	1	79	156	9	2
x4	1	20	236	24	1
x5	0	6	198	100	13

Table 2 Example. Reference data of the standard-agent A_0

Suppose that the PM has specified a certain chain of, increasingly challenging, binary-outcome goals

$$O_0 \preceq O_1 \preceq O_2 \preceq ... \preceq O_l \preceq ... \preceq \dots \preceq O_{L-1} \preceq O_L := O_{Full}, \tag{1}$$

which are hierarchically (i.e. Guttman like) ordered. Then, the (nominally recoded on $\{0, 1, \dots, L\}$) ordinal outcome indicator Y is defined so that the event occurrence "$Y \geq l$" identifies the achieving of the l-th scheduled goal $O_l := (Y \geq l)$, $l := 0, \dots, L$. Therefore, the pursued "full purpose" could be realized at different degree of achieving, from the "tautological" (i.e. alway achieved) goal $O_0 := (Y \geq 0)$ toward the final goal O_L. Let $\mathscr{P}^*$denote the population of the (real or perhaps virtual) individuals which are governed by the reference agent *(*e.g. a recognized "best practice" for standardization*)* A_0 (e.g. see table 2). Then, the criterion of intrinsic worthiness (see [3]) may be interpreted[1] on a goal-based probabilistic setup as follows.

For any actual individual i, having achieved goal O_{l-1} on chain of goals (1), the higher "the risk of failing the next goal O_l", referring such a risk on the population $\mathscr{P}^*$, the greater the "increase of worthiness", due to the performance of the agent which governs i "as if" i was in $\mathscr{P}^*$, whenever it actually achieves goal O_l.

[1] Consider hierarchical chain of goals *(1)*. Given that a certain goal O_{l-1} has been achieved, the greater the resistance, with reference to the evaluation framework, to also achieve the next pursued goal O_l*,* by continuing to improve, the greater the increment of value due to the intrinsic worthiness of who, effectively, is able to achieve it.

Thus, the $\mathscr{P}^*-$standardized, conditionally on status $x := X \in \{x_1, \ldots, x_R\}$, worthiness increase between any two adjacent levels of $Y := l \in (0,1,..,L)$ is provided[2] (for $l := 1,..,L$) by:

$$\omega^*_{l|x} := \Delta_{l-1} Val_{|x} := Val_{|x}(O_l) - Val_{|x}(O_{l-1}) =$$
$$= \varphi_l\Big(\frac{Pr\{Y = l-1|x;\, \mathscr{P}^*\}}{Pr\{Y \geq l-1|x;\, \mathscr{P}^*\}}\Big) = \varphi_l\Big(\frac{p_{l-1|x}}{p_{l-1|x} + p_{l|x} + \cdots + p_{L|x}}\Big) \geq 0 \qquad (2)$$

Here, continuous monotone functions $\varphi_l(.)$ (e.g. set here the identity) of the conditional probability rates may be chosen (see [3]) for specifying some characteristics (e.g. the additivity) of the scale. Formally re-interpreting "worthiness increases" as "utility increases", functionals of the "rank dependent expected utility", adhering to requirements of rational choices (e.g. see [2], pp. 559), leads to the following instance of conditional-expectation-based index[3]:

$$u :\longmapsto W[p_x[u];\, \omega^*_x, x] := \sum_{l:=1}^{L} \varphi_l\Big(\frac{Pr\{Y = l-1|\, x; \mathscr{P}^*\}}{Pr\{Y \geq l-1|\, x; \mathscr{P}^*\}}\Big) \cdot (1 - F_{Y|x}[p[u]](l)) \qquad (3)$$

Here, $F_{Y|x}[p]$ denotes the cumulative distribution such that $F_{Y|x}[p](l) = p_{0|x} + p_{1|x} + \cdots + p_{(l-1)|x}$. Thus, through $x \in \{x_1, \ldots, x_R\}$, it may be defined the global evaluation index: $u \longmapsto \sum_{r:=1}^{R} q_r \cdot W[p_{x_r}([u];\, \omega_{x_r}(\mathscr{P}^*)]$. It uses the actual agents data (e.g. see table 1)), standardized on the reference-agent's data (e.g. see table 2). Here, $q_r \geq 0$ ($\sum_{i:=1}^{R} q_r = 1$) weights[4] the reference domain for the status x_r.

2 Eliciting standardized worthiness increases

To justify differences in "worthiness increases" (2), through reference conditions $x := X \in \{x_1, \ldots, x_R\}$, the PM may adopt some "reference evaluation criterion" and working assumptions formally specified by means of a structural probabilistic model (4)-(7). Here[5], the conditional rates (1-v_{rl}) (which enter "worthiness increases"

[2] It is the worthiness credit which is gained by any social agent in improving the condition of a "*standard individual*", in the reference condition $x \in \{x_1, \ldots, x_R\}$, from the current level $(l-1)$ to the next l on the scale of Y which was constructed on goal chain (1).

[3] for any agent u, given x, it takes into input the distribution realized (e.g. see table 1), by the individuals that u governs in condition x, on the standardized worthiness-quantified levels of Y.

[4] these weights should represent the political relevancy of the "social reference domains" to the main aim of the PM.

[5] On the stratum of the n_r individuals in the condition x_r, the manifest outcome $(Y_{r0}, \ldots, Y_{rL})$ is distributed as a multinomial (eq.4) where the expectation-parameters $\psi_r := (\psi_{r0}, \psi_{r1}, \ldots, \psi_{rL})$ are normed, within the container Dirichlet model (eq. 5), on a set of constraints on the latent evolutionary processes undertaken the levels of outcome scale Y (eqs (6)-(7)).

(2)) are represented as latent parameters of interrelated evolutionary-processes behind the goals chain (1), which are driven by manifest conditions x up to hyper-parameters[6] $\gamma := (\mu_0, \delta, \beta^X)$ to be regulated. Then, the methodological question arises in automatic eliciting of values γ^* so that "worthiness increases" $\omega_{l|x}(\mathscr{P}^*; \gamma^*)$ enter evaluation indexes (3). To norm the model on the reference-agent's data table (2), recalling a "minimum information principle"[7], hyper-parameters γ may be regulated (e.g. see [5],[4]) to that value γ^* such that the *"residual from updating"*[8] $\| \; Vec \; (\; E(\Psi \mid y,x;\gamma,w) - E(\Psi \mid x;\gamma,\, w) \;) \; \|$ is minimized subject to specifications of constraints (6)-(7).

$$Y_r{:=}\{Y_{r0},...,Y_{rL}\}|\psi_r \overset{ind.}{\sim}_{r:=1,...,R} Mult(y_{r0},...,y_{rL};\psi_{r0},\psi_{r1},...,\psi_{rL},n_r) \tag{4}$$

$$\psi_r{:=}(\psi_{r0},\psi_{r1},...,\psi_{rL})|m_r,a_r \overset{ind.}{\sim}_{r:=1,...,R} Dirichlet(\psi_{r0},...,\psi_{rL};m_r,a_r) \tag{5}$$

$$m_r{:=}(m_{r0},\, m_{r1},...,m_{rL}),\; 0{<}m_{rl}{:=}E[\psi_{rl}]{<}1,\; \textstyle\sum_{s:=0}^{L} m_{rs}{=}1, \quad a_r{:=}w_r,\, w_r{>}0$$

$$v_{r1}{:=}\frac{m_{r0}}{m_{r0}+m_{r1}} = \frac{e^{\eta_{r1}}}{1+e^{\eta_{r1}}}, \tag{6}$$

$$v_{rl}{:=}\frac{m_{r0}+...+m_{r(l-1)}}{m_{r0}+m_{r1}+...+m_{rl}} = \frac{e^{\eta_{irl}}}{1+e^{\eta_{rl}}},$$

$$\cdots$$

$$v_{rL}{:=}\frac{m_{r0}+...+m_{r(L-1)}}{m_{r0}+m_{r1}+...+m_{rL}} = \frac{e^{\eta_{rL}}}{1+e^{\eta_{rL}}},$$

$$\eta_{rl}{=}\mu_0{+}\textstyle\sum_{s:=1}^{L} \delta_l \cdot I_{(s=l)} {+} \sum_{w:=2}^{R} \beta^X_{w(l-1)} \cdot I_{(X(r)=w)} \tag{7}$$

$$reference\, condition\, r{:=}1,...,R{:=}5;\; scale\, level\, transitions\, l{:=}1,...,L{:=}4$$

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[6] Hyper-parameters μ_0, δ represent, respectively, the common base-line and increments in the level scores of the scale; instead, the parameters of profileβ^X represent the crossed effects of condition and levels, in transition processes. Here, $I_{(.)}$ denote a binary indicator function.

[7] "The less a prior representation of knowledge is updated by current data, the more intrinsically it already was accounted for by the *intrinsic information* added by such data"

[8] Here, $E(\Psi \mid y,x;\gamma,w)$ denotes the predictive expectation of full parameter profile $\Psi := (\psi_1, .., \psi_R)$ which is updated by outcome y, whereas $E(\Psi \mid x;\gamma,w)$ is his non updated counterpart, over the design-point x

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