

# Chapter 1

## Statistical Analysis of Markov Switching DSGE Models

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**Abstract** We investigate statistical properties of Markov switching Dynamic Stochastic General Equilibrium (MS DSGE) models:  $L^2$  structure, stationarity, autocovariance function, and spectral density.

**Abstract** In questo lavoro si studiano le proprietà statistiche dei modelli Markov switching Dynamic Stochastic General Equilibrium (MS DSGE):  $L^2$  struttura, stazionarietà e le funzioni di autocovarianza e di densità spettrale.

**Key words:** Multivariate DSGE, State-Space models, Markov chains, changes in regime, autocovariance structure, spectral density function

### 1.1 Introduction

Dynamic Stochastic General Equilibrium (DSGE) models have recently gained a central role for analyzing the mechanisms of propagation of economic shocks. See, for example, Fernández–Villaverde *et al.* (2007) and Giacomini (2013). However, empirical work has shown that DSGE models have failed to fit the data very well over a long period of time. In fact, the changes of parameters require the economists to re–estimate them. This observation leads to Markov switching (MS) DSGE models, that is, DSGE models in which the coefficient parameters are assumed to depend on the state of an unobserved Markov chain.

Since the influential work of Hamilton (1989, 1990), Markov switching models are widely used to capture the business cycle regime shifts typically observed in

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economic data. See also Hamilton (1994, §22), and Hamilton (2005) for more recent references. Markov switching VARMA models have been studied by many authors. For information concerning the stationarity, estimation, consistency, asymptotic variance and model selection of MS VARMA models see Hamilton (cited above), Krolzig (1997), Francq and Zakoian (2001), Yang (2000), Zhang and Stine (2001), and Cavicchioli (2014a). See also Cavicchioli (2014b), where explicit matrix expressions for the maximum likelihood estimator of the parameters in MS VAR(CH) models have been derived. Higher-order moments and asymptotic Fisher information matrix of MS VARMA models are provided in Cavicchioli (2017a) and (2017b), respectively.

The first purpose of the present paper is to investigate the  $L^2$ -structure of the MS DSGE models. We derive stationarity conditions, compute explicitly the autocovariance function for such models, and give stable VARMA representation of them. A second goal of the paper is to propose a tractable method to derive the spectral density in a matrix closed-form of MS DSGE models. Then we illustrate some statistical properties of their spectral representation.

## 1.2 Stationarity of MS DSGE models

Let us consider the following Markov switching DSGE (in short, MS DSGE) model

$$\mathbf{x}_t = \mathbf{A}_{s_t} \mathbf{x}_{t-1} + \mathbf{B}_{s_t} \mathbf{w}_t \quad (1.1)$$

$$\mathbf{y}_t = \mathbf{C}_{s_t} \mathbf{x}_{t-1} + \mathbf{D}_{s_t} \mathbf{w}_t \quad (1.2)$$

where  $\mathbf{x}_t$  is an  $n \times 1$  vector of possibly unobserved state variables,  $\mathbf{y}_t$  is the  $k \times 1$  vector of observable variables, and  $\mathbf{w}_t \sim \text{NID}(0, \mathbf{I}_m)$  is an  $m \times 1$  vector of economic shocks. The matrices  $\mathbf{A}_{s_t} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{B}_{s_t} \in \mathbb{R}^{n \times m}$ ,  $\mathbf{C}_{s_t} \in \mathbb{R}^{k \times n}$  and  $\mathbf{D}_{s_t} \in \mathbb{R}^{k \times m}$  are real random matrices.

The process  $(s_t)$  is a homogeneous stationary irreducible and aperiodic Markov chain with finite state-space  $\mathcal{E} = \{1, \dots, d\}$ . Let  $\pi(i) = \Pr(s_t = i)$  denote the ergodic probabilities, which are positive. Let  $p(i, j) = \Pr(s_t = j | s_{t-1} = i)$  be the stationary transition probabilities. Then the  $d \times d$  matrix  $\mathbf{P} = (p(i, j))$  is called the *transition probability matrix*. The process  $(s_t)$  is independent of  $(\mathbf{w}_t)$ .

In order to investigate the stationarity properties of the process  $(\mathbf{y}_t)$ , we use the following vectorial representation of the MS DSGE model:

$$\mathbf{z}_t = \Phi_{s_t} \mathbf{z}_{t-1} + \Psi_{s_t} \mathbf{w}_t \quad (1.3)$$

where

$$\mathbf{z}_t = \begin{pmatrix} \mathbf{x}_t \\ \mathbf{y}_t \end{pmatrix} \quad \Phi_{s_t} = \begin{pmatrix} \mathbf{A}_{s_t} & 0 \\ \mathbf{C}_{s_t} & 0 \end{pmatrix} \quad \Psi_{s_t} = \begin{pmatrix} \mathbf{B}_{s_t} \\ \mathbf{D}_{s_t} \end{pmatrix}.$$

Then  $\mathbf{z}_t$  is  $p \times 1$ ,  $\Phi_{s_t}$  is  $p \times p$  and  $\Psi_{s_t}$  is  $p \times m$ , where  $p = n + k$ . Let  $\Phi_i$  and  $\Psi_i$  be the matrices obtained by replacing  $s_t$  by  $i$  in  $\Phi_{s_t}$  and  $\Psi_{s_t}$ , for  $i = 1, \dots, d$ . Let us consider the following matrices:

$$\mathbb{P}_{\Phi \otimes \Phi} = \begin{pmatrix} p(1,1)\{\Phi_1 \otimes \Phi_1\} & p(2,1)\{\Phi_1 \otimes \Phi_1\} & \cdots & p(d,1)\{\Phi_1 \otimes \Phi_1\} \\ p(1,2)\{\Phi_2 \otimes \Phi_2\} & p(2,2)\{\Phi_2 \otimes \Phi_2\} & \cdots & p(d,2)\{\Phi_2 \otimes \Phi_2\} \\ \vdots & \vdots & \ddots & \vdots \\ p(1,d)\{\Phi_d \otimes \Phi_d\} & p(2,d)\{\Phi_d \otimes \Phi_d\} & \cdots & p(d,d)\{\Phi_d \otimes \Phi_d\} \end{pmatrix}$$

and

$$\Pi_{\Psi \otimes \Psi} := \begin{pmatrix} \pi(1)\{\Psi_1 \otimes \Psi_1\} \\ \pi(2)\{\Psi_2 \otimes \Psi_2\} \\ \vdots \\ \pi(d)\{\Psi_d \otimes \Psi_d\} \end{pmatrix}.$$

Then  $\mathbb{P}_{\Phi \otimes \Phi}$  is  $(dp^2) \times (dp^2)$  and  $\Pi_{\Psi \otimes \Psi}$  is  $(dp^2) \times m^2$ . Consider the top Lyapunov exponent

$$\gamma_{\Phi} = \inf_{t \in \mathbb{N}} \left\{ E \frac{1}{t} \log_e \|\Phi_{s_t} \Phi_{s_{t-1}} \cdots \Phi_{s_1}\| \right\}. \quad (1.4)$$

**Theorem 1.** *If  $\gamma_{\Phi} < 0$ , then the process  $(\mathbf{y}_t)$  is the unique strictly stationary solution of the MS DSGE model in (1.1) and (1.2).*

**Theorem 2.** *If  $\rho(\mathbb{P}_{\Phi \otimes \Phi}) < 1$ , where  $\rho(\cdot)$  denotes the spectral radius, then the process  $(\mathbf{y}_t)$  is the unique nonanticipative second-order stationary solution of the MS DSGE model in (1.1) and (1.2).*

### 1.3 Autocovariance structure and spectral analysis

Let  $(\mathbf{z}_t)$  (and  $(\mathbf{y}_t)$ ) be second-order stationary. For every  $h$ , define

$$W(h) = \Pi_{E(\mathbf{z}_t \mathbf{z}'_{t-h})} := \begin{pmatrix} \pi(1) E(\mathbf{z}_t \mathbf{z}'_{t-h} | s_t = 1) \\ \pi(2) E(\mathbf{z}_t \mathbf{z}'_{t-h} | s_t = 2) \\ \vdots \\ \pi(d) E(\mathbf{z}_t \mathbf{z}'_{t-h} | s_t = d) \end{pmatrix} \in \mathbb{R}^{(dp) \times p}.$$

For  $h = 0$ , we prove that

$$\text{vec } W(0) = \mathbb{P}_{\Phi \otimes \Phi} \text{vec } W(0) + \Pi_{\Psi \otimes \Psi} \text{vec}(\mathbf{I}_m) \quad (1.5)$$

where  $\mathbf{I}_m$  is the identity matrix of order  $m$ . Let  $\mathbb{P}_{\Phi}$  be the  $(dp) \times (dp)$  matrix obtained by replacing  $\Phi_i \otimes \Phi_i$  by  $\Phi_i$ , for  $i = 1, \dots, d$ , in the definition of  $\mathbb{P}_{\Phi \otimes \Phi}$ . For  $h \geq 0$ , we prove that

$$W(h) = \mathbb{P}_{\Phi}^h W(0). \quad (1.6)$$

**Theorem 3.** *Suppose that  $\rho(\mathbb{P}_{\Phi \otimes \Phi}) < 1$ . Then the autocovariance function of the process  $(\mathbf{y}_t)$  defined by the MS DSGE model in (1.1) and (1.2) is given by*

$$\Gamma_{\mathbf{y}}(h) = \mathbf{f}(\mathbf{e}' \otimes \mathbf{I}_p) \mathbb{P}_{\Phi}^h W(0) \mathbf{f}'$$

for every  $h \geq 0$ , where

$$\begin{aligned} \text{vec } W(0) &= \left( \mathbf{I}_{dp^2} - \mathbb{P}_{\Phi \otimes \Phi} \right)^{-1} \Pi_{\Psi \otimes \Psi} \text{vec } \mathbf{I}_m \\ \mathbf{f} &= (0_{k \times n} \quad \mathbf{I}_k) \in \mathbb{R}^{k \times p} \quad \mathbf{e} = (1 \dots 1)' \in \mathbb{R}^d. \end{aligned}$$

Theorem 3 allows to obtain a stable VARMA representation for any second-order stationary process  $(\mathbf{y}_t)$  driven by an MS DSGE model.

**Theorem 4.** *Suppose that  $\rho(\mathbb{P}_{\Phi \otimes \Phi}) < 1$ . The spectral density matrix of the process  $(\mathbf{y}_t)$  driven by the MS DSGE in (1.1) and (1.2) is given by*

$$\mathbf{F}_{\mathbf{y}}(\omega) = \mathbf{f}(\mathbf{e}' \otimes \mathbf{I}_p) [-\mathbf{I}_{dp} + 2\text{Re}Y(\omega)] W(0) \mathbf{f}'$$

where

$$Y(\omega) = (\mathbf{I}_{dp} - \mathbb{P}_{\Phi} e^{-i\omega})^{-1}$$

is a  $(dp) \times (dp)$  complex matrix and  $\text{Re}Y(\omega)$  denotes the real part of  $Y(\omega)$ .

Theorems 1–4 are the main results proved in Cavicchioli (2018). Examples and numerical applications complete the mentioned paper.

## 1.4 Reference

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