Time-varying long-memory processes

Processi a memoria lunga con parametro frazionario variabile nel tempo

Luisa Bisaglia and Matteo Grigoletto

Abstract In this work we propose a new class of long-memory models with time-varying fractional parameter. In particular, the dynamics of the long-memory coefficient, d, is specified through a stochastic recurrence equation driven by the score of the predictive likelihood, as suggested by Creal *et al.* (2013) and Harvey (2013).

Key words: long-memory, GAS model, time-varying parameter

1 Introduction

Long-memory processes have proved to be useful tools in the analysis of many empirical time series. These series present the property that the autocorrelation function at large lags decreases to zero like a power function rather than exponentially, so that the correlations are not summable.

One of the most popular processes that takes into account this particular behavior of the autocorrelation function is the AutoRegressive Fractionally Integrated Moving Average process (ARFIMA(p,d,q)), independently introduced by Granger and Joyeux (1980) and Hosking (1981). This process generalizes the ARIMA(p,d,q) process by relaxing the assumption that d is an integer.

The ARFIMA(p,d,q) process, Y_t, is defined by the difference equation

$$\Phi(B)(1-B)^d(Y_t-\mu) = \Theta(B)\varepsilon_t,$$

where $\varepsilon_t \sim WN(0,\sigma^2)$, and $\Phi(\cdot)$ and $\Theta(\cdot)$ are polynomials in the backward shift operator B of degrees p and q, respectively. Furthermore, $(1-B)^d = \sum_{j=0}^{\infty} \pi_j B^j$,

Luisa Bisaglia
Dept. of Statistical Science, University of Padova, e-mail: luisa.bisaglia@unipd.it

Matteo Grigoletto
Dept. of Statistical Science, University of Padova, e-mail: matteo.grigoletto@unipd.it

with $\pi_j = \Gamma(j-d)/[\Gamma(j+1)\Gamma(-d)]$, where $\Gamma(\cdot)$ denotes the gamma function. When the roots of $\Phi(B) = 0$ and $\Theta(B) = 0$ lie outside the unit circle and $| d |< 0.5$, the process is stationary, causal and invertible. We will assume these conditions to be satisfied.

When $d \in (0, 0.5)$ the autocorrelation function of the process decays to zero hyperbolically at a rate $O(k^{2d-1})$, where k denotes the lag. In this case we say that the process has a long-memory behavior. When $d \in (-0.5, 0)$ the process is said to have intermediate memory.

If $p = q = 0$, the process $\{Y_t,\ t = 0, \pm 1, \ldots\}$ is called Fractionally Integrated Noise, $FI(d)$. In the following we will concentrate on $FI(d)$ processes with $d \in (-0.5, 0.5)$.

Several papers have addressed the detection of breaks in the order of fractional integration. Some of these works allowed for just one unknown breakpoint (see, for instance, Berand and Terrin, 1996; Yamaghuchi, 2011). Others treated the number of breaks as well as their timing as unknown (Ray and Tsay, 2002; Hassler and Meller, 2014). Boutahar *et al.* (2008) generalize the standard long memory modeling by assuming that the long memory parameter d is stochastic and time-varying. The authors introduce a STAR process, characterized by a logistic function, on this parameter and propose an estimation method for the model. Finally, Roueff and von Sachs (2011) take into account the time-varying feature of long-memory parameter d using the wavelets approach.

Our approach is completely different because we allow the long memory parameter d to vary at each time t. Moreover, our approach is based on the theory of Generalized Autoregressive Score (GAS) models. In particular, the peculiarity of our approach is that the dynamics of the long-memory parameter is specified through a stochastic recurrence equation driven by the score of the predictive likelihood. In this way we are able to take into account also smooth changes of the long-memory parameter.

2 GAS model

To allow for time-varying parameters, Creal *et al.* (2013) and Harvey (2013) proposed an updating equation where the innovation is given by the score of the conditional distribution of the observations (GAS models). The basic framework is the following. Consider a time series $\{y_1, \cdots, y_n\}$ with time-t observation density $p(y_t \mid \psi_t)$, where $\psi_t = (f_t, \theta)$ is the parameter vector, with f_t representing the time-varying parameter(s) and θ the remaining fixed coefficients.

In time series the likelihood function can be written via prediction errors as:

$$\mathscr{L}(y, \psi) = p(y_1; \psi_1) \prod_{t=2}^{n} p(y_t \mid y_1, \cdots, y_{t-1}; \psi_1, \cdots, \psi_t) \ .$$

Thus, the t-th contribution to the log-likelihood is:

$$l_t = \log p(y_t \mid y_1, \cdots, y_{t-1}; f_1, \cdots, f_t; \theta) = \log p(y_t \mid y_1, \cdots, y_{t-1}; f_t; \theta) \ ,$$

where we assume that $f_1, \cdots, f_t$ are known (because they are realized).

The parameter value for the next period, f_{t+1}, is determined by an autoregressive updating function that has an innovation equal to the score of l_t with respect to f_t. In particular, we can assume that:

$$f_{t+1} = \omega + \beta f_t + \alpha s_t \ ,$$

where the innovation s_t is given by

$$s_t = S_t \cdot \nabla_t \ ,$$

with

$$\nabla_t = \frac{\partial \log p(y_t \mid y_1, \cdots, y_{t-1}; f_t, \theta)}{\partial f_t} \tag{1}$$

and

$$S_t = \mathscr{I}_{t-1}^{-1} = -E_{t-1} \left[\frac{\partial^2 \log p(y_t \mid y_1, \cdots, y_{t-1}; f_t, \theta)}{\partial f_t \partial f_t^{'}} \right]^{-1} \ . \tag{2}$$

By determining f_{t+1} in this way, we obtain a recursive algorithm for the estimation of time-varying parameters.

3 TV-FI(d) model

In this section, we extend the class of $FI(d)$ models, by allowing the long-memory parameter d to change over time. The dynamics of the time-varying coefficient d_t is specified in the GAS framework outlined above.

The $TV-FI(d)$ model is described by the following equations:

$$(1-B)^{d_t} \ y_t = \varepsilon_t \ ,$$

$$d_{t+1} = \omega + \beta d_t + \alpha s_t \ , \tag{3}$$

where $\varepsilon_t \sim iid\mathscr{N}(0, \sigma^2)$, and $s_t = S_t \nabla_t$ with S_t and ∇_t defined below.

To calculate the score of the log-likelihood it is preferable to consider the use of autoregressive representation (see, for instance, Palma, 2007):

$$(1-B)^{d_t} y_t = y_t + \sum_{j=1}^{\infty} \pi_j(d_t) y_{t-j} = \varepsilon_t \ ,$$

where

$$\pi_j(d_t) = \prod_{k=1}^{j} \frac{k-1-d_t}{k} = -\frac{d_t \Gamma(j-d_t)}{\Gamma(1-d_t)\Gamma(j+1)} = \frac{\Gamma(j-d_t)}{\Gamma(-d_t)\Gamma(j+1)} \; .$$

In practice, only a finite number n of observations is available. Therefore, we use the approximation

$$y_t = -\pi_1(d_t)\, y_{t-1} - \pi_2(d_t)\, y_{t-2} - \cdots - \pi_m(d_t)\, y_{t-m} + \varepsilon_t \; ,$$

with $m < n$. Then, the t-th contribution, $t = 1, \ldots, n$, to the log-likelihood is:

$$l_t(d_t, \sigma^2) = c - \log(\sigma^2) - \frac{1}{\sigma^2} \left(y_t + \sum_{j=1}^{t-1} \pi_j(d_t) y_{t-j} \right)^2$$

where c is a constant and the corresponding score of the predictive likelihood, see equation (1), becomes

$$\nabla_t = -\frac{1}{\sigma^2} \left(y_t + \sum_{j=1}^{t-1} \pi_j(d_t)\, y_{t-j} \right) \left(\sum_{j=1}^{t-1} \nu_j(d_t)\, y_{t-j} \right) \; , \tag{4}$$

where

$$\nu_j(d_t) = \frac{\partial \pi_j(d_t)}{\partial d_t} = \pi_j(d_t) \left(-\Psi(j-d_t) + \Psi(1-d_t) + \frac{1}{d_t} \right) \; , \tag{5}$$

with $\Psi(\cdot)$ representing the digamma function. Finally, we find that S_t in equation (2) is

$$S_t = \sigma^2 \cdot \left(\sum_{j=1}^{t-1} \nu_j(d_t)\, y_{t-j} \right)^{-2} \; .$$

4 Some Monte Carlo results

We simulated $y_1, \ldots, y_n$ from a process

$$(1-B)^{d_t} y_t = \varepsilon_t \; , \tag{6}$$

where $\varepsilon_t \sim iid\mathcal{N}(0, \sigma^2)$, and d_t is defined by

$$d_t = 0.1 + 0.3 \; \frac{t}{n} \tag{7}$$

or

$$d_t = 0.1 + 0.3 \; \Phi \left(\frac{t - n/2}{3\sqrt{n}} \right) \; , \tag{8}$$

with $\Phi(\cdot)$ indicating the standard Gaussian distribution function.

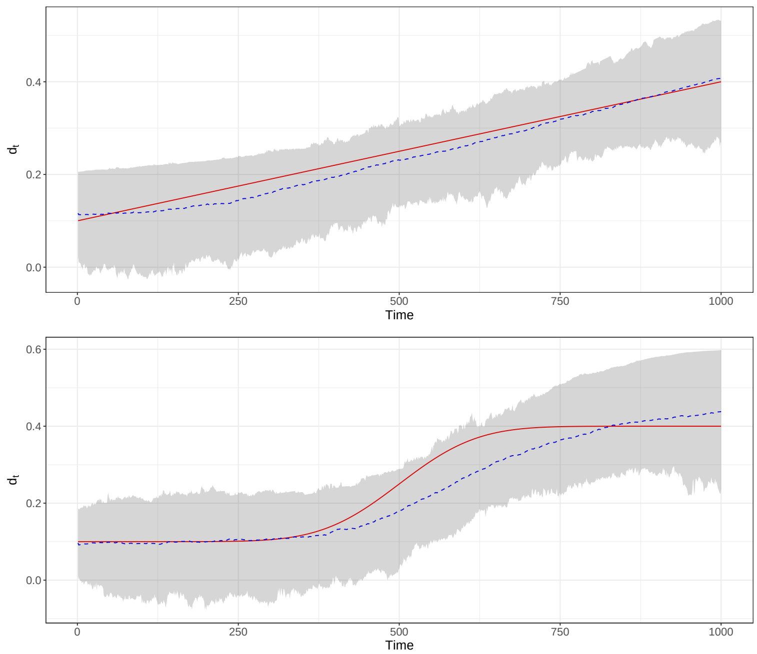

Fig. 1 Result of 200 Monte Carlo simulations, where a time variable fractional parameter (solid line) is estimated with a TV-I(d) model. The dashed line represent the average estimates, while the gray band shows the empirical 95% intervals.

The evolution of d_t was then estimated using the TV-FI(d) model introduced above. It should be noted that in GAS models the scaling defined by (2) is often replaced by S_t^{γ}, for some suitable γ. We found results (Creal *et al.*, 2013) to be more stable with $\gamma = 0.5$. Also, GAS models can easily be accommodated in order to include a link function $\Lambda(\cdot)$, typically with the objective to constrain the parameter of interest to vary in some region. We used

$$d_t = \Lambda(g_t) = a + (b-a)\,\frac{e^{g_t}}{1+e^{g_t}}\ ,$$

so that $d_t \in (a,b)$, while $g_t \in \mathbb{R}$. Recursion (3) is then defined in terms of g_t, with (4) and (5) easily adjusted for the reparametrization.

It should be remarked that d_0, the value of the fractional at time 0, is necessary to define the likelihood. In the following, we treat d_0 as a parameter to be estimated along with the others.

We obtained 200 Monte Carlo replications from the process defined by (6), and (7) or (8), setting $n = 1000$ and $\sigma = 2$.

For each replication, the TV-FI(d) model was estimated by maximum likelihood, setting $(a,b) = (-0.4, 0.6)$ and $\omega = 0$, while estimating $(d_0, \alpha, \beta, \sigma)$.

Simulation results are shown in Figure 1. The solid line shows the true evolution of d_t, while the dashed line is its estimate, averaged over the Monte Carlo replications. The gray band represents the empirical 95% intervals.

References

1. Beran J. and Terrin N.: Testing for a change of the long-memory parameter. Biometrika, **83**, 627–638 (1996).
2. Boutahar M., Dufrénot G. and Péguin-Feissolle A.: A Simple Fractionally Integrated Model with a Time-varying Long Memory Parameter d_t. Computational Economics, **31**, 225–241 (2008).
3. Creal, Drew D., Koopman S.I. and Lucas A.: Generalized Autoregressive Score Models with Applications. Journal of Applied Econometrics, **28**, 777–795 (2013).
4. Granger, C.W.J. and Joyeux, R.: An introduction to long-range time series models and fractional differencing. Journal of Time Series Analysis, **1**, 15–30 (1980).
5. Harvey, A.C.: Dynamic Models for Volatility and Heavy Tails: With Applications to Financial and Economic Time Series. Econometric Series Monographs. Cambridge University Press (2013).
6. Hassler, U. and Meller, B.: Detecting multiple breaks in long memory the case of U.S. inflation. Empirical Economics, **46**, 653–680 (2014).
7. Hosking, J.R.M.: Fractional differencing. Biometrika, **68**, 165–176 (1981).
8. Palma, W.: Long-memory Time Series. Wiley, New Jersey (2007).
9. R Core Team. R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria, (2015). URL https://www.R-project.org/.
10. Ray, B.K. and Tsay R.S.: Bayesian methods for change-point detection in long-range dependent processes. Journal of Time Series Analysis, **23**, 687–705 (2002).
11. Roueff, F. and von Sachs, R.: Locally stationary long memory estimation. Stochastic Processes and their Applications, **121**, 813–844 (2011).
12. Yamaguchi, K.: Estimating a change point in the long memory parameter. Journal of Time Series Analysis, **32**, 304–314 (2011).