Conditional Quantile-Located VaR

Valore a rischio condizionato ai quantili

Giovanni Bonaccolto, Massimiliano Caporin and Sandra Paterlini

Abstract The Conditional Value-at-Risk (CoVaR) has been proposed by [1] to measure the impact of a company in distress on the Value-at-Risk (VaR) of the financial system. We propose here an extension of the CoVaR, that is, the Conditional Quantile-Located VaR (QL-CoVaR), that better deals with tail events, when spillover effects impact the stability of the entire system. In fact, the QL-CoVaR is estimated by assuming that the financial system and the individual companies simultaneously lie in the left tails of their distributions.

Abstract *Il valore a rischio condizionato (CoVaR) é stato introdotto da [1] per quantificare l'impatto di una societá in fase di stress sul valore a rischio (VaR) del sistema finanziario. Nel presente lavoro proponiamo un'estensione del VaR (QL-CoVaR), che meglio si adatta agli eventi estremi, quando il rischio di contagio impatta sulla stabilitá dell'intera economia. Infatti, il QL-CoVaR é stimato assumendo che il sistema finanziario e le singole societá sono simultaneamente poste sulle code sinistre delle loro distribuzioni.*

Key words: CoVaR, systemic risk, quantile-on-quantile relationships

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1 Methods

We first introduce the Conditional Value-at-Risk (CoVaR) proposed by [1]. Then, we provide the details about the Conditional Quantile-Located Value-at-Risk (QL-CoVaR). Let y_t and $x_{i,t}$ be the returns of the financial system and of the i-th financial company at time t, respectively, for $i = 1,...,N$ and $t = 1,...,T$. Let $Q_\tau(x_{i,t}|\mathbf{I}_{t-1})$ denotes the τ-th quantile of $x_{i,t}$, for $\tau \in (0,1)$, conditional to the information set $\mathbf{I}_{t-1}$, where $\mathbf{I}_{t-1} = (y_{t-1}, x_{i,t-1}, m_{t-1})$ with m_{t-1} being a control variable at time $t-1$. Similarly, $Q_\theta(y_t|\mathbf{I}_{t-1}, x_{i,t})$ is the θ-th quantile of y_t conditional to the information set available at $t-1$ as well as to the return of the i-th company observed at time t, for $\theta \in (0,1)$. For simplicity, we set $Q_\tau(x_{i,t}|\mathbf{I}_{t-1}) \equiv Q_\tau(x_{i,t})$ and $Q_\theta(y_t|\mathbf{I}_{t-1}, x_{i,t}) \equiv Q_\theta^{(i)}(y_t)$; θ and τ take low values, typically in the interval $(0, 0.05)$. The CoVaR introduced by [1] is then estimated from the quantile regression models (see [4]):

$$Q_\tau(x_{i,t}) = \alpha_\tau^{(i)} + \beta_\tau^{(i)} m_{t-1}, \tag{1}$$

$$Q_\theta^{(i)}(y_t) = \delta_\theta^{(i)} + \lambda_\theta^{(i)} x_{i,t} + \gamma_\theta^{(i)} m_{t-1}. \tag{2}$$

Let $\widehat{Q}_\tau(x_{i,\tau}) = \widehat{\alpha}_\tau^{(i)} + \widehat{\beta}_\tau^{(i)} m_{t-1}$ be the estimated τ-th quantile of $x_{i,t}$, it is possible to compute the CoVaR at the distress and at the median state of the conditioning company, respectively, as follows:

$$CoVaR_{t,\theta,\tau}^{(i)} = \widehat{\delta}_\theta^{(i)} + \widehat{\lambda}_\theta^{(i)} \widehat{Q}_\tau(x_{i,t}) + \widehat{\gamma}_\theta^{(i)} m_{t-1}, \tag{3}$$

$$CoVaR_{t,\theta,1/2}^{(i)} = \widehat{\delta}_\theta^{(i)} + \widehat{\lambda}_\theta^{(i)} \widehat{Q}_{1/2}(x_{i,t}) + \widehat{\gamma}_\theta^{(i)} m_{t-1}, \tag{4}$$

and compute the ΔCoVaR to quantify the marginal contribution of the i-th company to the systemic risk (see [1]). Note that $CoVaR_{t,\theta,1/2}^{(i)}$ is always parameterized to the median state of the i-th conditioning company. Hence, we can omit the level 1/2 as subscript of the ΔCoVaR measure as follows:

$$\Delta CoVaR_{t,\theta,\tau}^{(i)} = CoVaR_{t,\theta,\tau}^{(i)} - CoVaR_{t,\theta,1/2}^{(i)} = \widehat{\lambda}_\theta^{(i)} \left[\widehat{Q}_\tau(x_{i,t}) - \widehat{Q}_{1/2}(x_{i,t}) \right]. \tag{5}$$

For simplicity, we set $\theta = \tau$ and, then, $\Delta CoVaR_{t,\theta,\tau}^{(i)} \equiv \Delta CoVaR_{t,\tau}^{(i)}$. It is important to highlight that the parameters in (2) and the coefficients in (3) are functions of θ only, neglecting the role of τ. Therefore, the estimation process behind (3) depends on $x_{i,t}$ and not on $Q_\tau(x_{i,t})$. In contrast, we estimate the parameters in (2) assuming that the financial system and the i-th company simultaneously lie in the left tails of their distributions. We then take into account the impact exerted by $x_{i,t}$—in the neighbourhood of its τ-th quantile—on $\widehat{Q}_\theta^{(i)}(y_t)$. This allows us to increase the distress degree in the connections between the individual companies and the system to make our risk measure more sensitive to extreme events. The model we propose is defined as follows:

$$Q^{(i)}_{\theta,\tau}(y_t) = \delta^{(i)}_{\theta,\tau} + \lambda^{(i)}_{\theta,\tau} x_{i,t} + \gamma^{(i)}_{\theta,\tau} m_{t-1}, \tag{6}$$

where the parameters have both θ and τ as subscripts, as they depend on the quantiles levels of both y_t and $x_{i,t}$.

In fact, the unknown parameters in (6) are estimated from the following minimization problem:

$$\underset{\delta^{(i)}_{\theta,\tau},\lambda^{(i)}_{\theta,\tau},\gamma^{(i)}_{\theta,\tau}}{\arg\min} \sum_{t=1}^{T} \rho_\theta \left[y_t - \delta^{(i)}_{\theta,\tau} - \lambda^{(i)}_{\theta,\tau} x_{i,t} - \gamma^{(i)}_{\theta,\tau} m_{t-1} \right] K \left(\frac{\widehat{F}_{t|t-1}(x_{i,t}) - \tau}{h} \right), \tag{7}$$

where $\rho_\theta(e) = e(\theta - \mathbf{1}_{\{e<0\}})$ is the asymmetric loss function used in the quantile regression method by [4]; $\mathbf{1}_{\{\cdot\}}$ is an indicator function, taking the value of 1 if the condition in $\{\cdot\}$ is satisfied, the value of 0 otherwise; $K(\cdot)$ is the kernel function, with bandwidth h, whereas $\widehat{F}_{t|t-1}(x_{i,t})$ is the empirical conditional quantile of $x_{i,t}$. [5] used a similar approach to estimate the relations in quantiles between oil prices and stock returns.

In contrast to [5], we estimate $\widehat{F}_{t|t-1}(x_{i,t})$ dynamically using a rolling window procedure. For each window, we estimate a large set of $x_{i,t}$ quantiles in the support $\tau \in (0,1)$ from the quantile regression model (1), using the method proposed by [2] to ensure the monotonicity of the multiple quantiles for $\tau \in (0,1)$. Then, we linearly interpolate the set of quantiles to obtain the conditional distribution of $x_{i,t}$ at time t, denoted as $\widehat{F}(x_{i,t}|m_{t-1})$. Finally, we recover $\widehat{F}_{t|t-1}(x_{i,t})$, as the probability level, extrapolated from $\widehat{F}(x_{i,t}|m_{t-1})$, corresponding to the realization $x_{i,t}$.

From the method described above, we then compute the QL-CoVaR at the τ-th level as follows:

$$QL\text{-}CoVaR^{(i)}_{t,\theta,\tau} = \widehat{\delta}^{(i)}_{\theta,\tau} + \widehat{\lambda}^{(i)}_{\theta,\tau} \widehat{Q}_\tau(x_{i,t}) + \widehat{\gamma}^{(i)}_{\theta,\tau} m_{t-1}, \tag{8}$$

where $\widehat{Q}_\tau(x_{i,t}) = \widehat{\alpha}^{(i)}_\tau + \widehat{\beta}^{(i)}_\tau m_{t-1}$.

Then, given $\theta = \tau$, and evaluating the model also for $\tau = 1/2$, we define the ΔQL-CoVaR as:

$$\begin{aligned}\Delta QL\text{-}CoVaR^{(i)}_{t,\tau} &= QL\text{-}CoVaR^{(i)}_{t,\theta,\tau} - QL\text{-}CoVaR^{(i)}_{t,\theta,1/2} = \widehat{\delta}^{(i)}_{\theta,\tau} - \widehat{\delta}^{(i)}_{\theta,1/2} \\ &+ \widehat{\lambda}^{(i)}_{\theta,\tau} \left[\widehat{Q}_\tau(x_{i,t}) - \widehat{Q}_{1/2}(x_{i,t}) \right] + (\widehat{\lambda}^{(i)}_{\theta,\tau} - \widehat{\lambda}^{(i)}_{\theta,1/2}) \widehat{Q}_{1/2}(x_{i,t}) \\ &+ (\widehat{\gamma}^{(i)}_{\theta,\tau} - \widehat{\gamma}^{(i)}_{\theta,1/2}) m_{t-1}.\end{aligned} \tag{9}$$

It is important to highlight that $\Delta QL\text{-}CoVaR^{(i)}_{t,\tau}$ includes more components than $\Delta CoVaR^{(i)}_{t,\tau}$ in (5), as the coefficients in (9) also depend on the state of the i-th company. We then have further information about the relationships between the financial system and the individual companies when we focus on the left tails of their distributions. We compute the standard errors of the ΔCoVaR and the ΔQL-CoVaR coefficients using the bootstrap approach (see, e.g., [3]).

2 Empirical results

We implement the methods discussed in Section 1 on the daily returns of 1,155 U.S. financial institutions (952 banks and 203 insurance companies) in the period between October 10, 2000 and July 31, 2015, for a total of 3,864 days.[1] We note that some of the companies enter the dataset after October 10, 2000, whereas others exit before July 31, 2015. We estimate the models described in Section 1 for each of the financial companies for which we have at least 200 observations, resulting in 1,030 companies. We also build an index providing the return of the financial system (y_t) from the returns of the 1,155 financial institutions included in our dataset, weighted by their market values. As for m_t, we use the first principal component of variables that are related to bond, equity and real estate markets: i) the CBOE Volatility Index (VIX); ii) the liquidity spread (LS), computed as the difference between the three-month collateral repo rate and the three-month bill rate; iii) the change in the three-month Treasury bill rate (TB); iv) the change in the slope of the yield curve (YC), computed as the spread between the ten-year Treasury rate and the three-month bill rate; v) the change in the credit spread between BAA-rated bonds and the Treasury rate (CS), both with the ten years maturity; vi) the daily equity market return (EM); vii) the excess return of the real estate sector over the market return (RE).[2] In particular, we checked that the first principal component (m_t) of the variables listed above explains 96.50% of the variability in the data.

Table 1 Estimation of $Q_{\theta}^{(i)}(y_t) = \delta_{\theta}^{(i)} + \lambda_{\theta}^{(i)} x_{i,t} + \gamma_{\theta}^{(i)} m_{t-1}$

	$\theta = 0.01$					$\theta = 0.05$				
COEF	5P	MED	95P	IQR	PS	5P	MED	95P	IQR	PS
δ_θ	-0.042	-0.031	-0.021	0.012	99.90	-0.027	-0.019	-0.014	0.007	99.61
λ_θ	-0.033	0.117	0.536	0.236	45.15	-0.006	0.112	0.561	0.276	57.09
$100 \times \gamma_\theta$	-0.295	-0.197	-0.074	0.107	88.45	-0.202	-0.128	-0.079	0.064	95.24

The table reports the summary statistics of the CoVaR's parameters estimated for the N financial companies included in our dataset. The estimates are obtained using two quantile levels—θ. In each panel, from left to right, we report the following descriptive statistics of the coefficients: the 5-th percentile (5P), the median (MED), the 95-th percentile (3Q), the interquartile range (IQR) and the percentage of times, out of N, in which they are statistically significant at the 5% confidence level (PS).

We estimate the CoVaR and the QL-CoVaR using two quantile levels—$\theta = \tau = 0.01$ and $\theta = \tau = 0.05$. As for the estimation of the QL-CoVaR parameters, we use the Gaussian kernel as $F(\cdot)$, with $h = 0.15$.[3] On the basis of the empirical set-up

[1] The data are recovered from Thomson Reuters Datastream.

[2] The control variables listed in i)—v) are taken from Thomson Reuters Datastream, whereas EM and RE are recovered from the industry portfolios built by Kenneth R. French, available at *http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html*.

[3] We used other values of h for a robustness check, that is, $h = \{0.10, 0.20\}$. We checked that the results obtained with different h values are qualitative similar and are available upon request.

Table 2 Estimation of $Q^{(i)}_{\theta,\tau}(y_t) = \delta^{(i)}_{\theta,\tau} + \lambda^{(i)}_{\theta,\tau} x_{i,t} + \gamma^{(i)}_{\theta,\tau} m_{t-1}$ and $Q^{(i)}_{\theta,1/2}(y_t) = \delta^{(i)}_{\theta,1/2} + \lambda^{(i)}_{\theta,1/2} x_{i,t} + \gamma^{(i)}_{\theta,1/2} m_{t-1}$

COEF	5P	MED	95P	IQR	PS	5P	MED	95P	IQR	PS
	$\theta = \tau = 0.01$					$\theta = \tau = 0.05$				
$\delta_{\theta,\tau}$	-0.049	-0.029	-0.016	0.014	95.05	-0.030	-0.018	-0.010	0.007	95.44
$\delta_{\theta,1/2}$	-0.041	-0.028	-0.019	0.011	99.22	-0.026	-0.018	-0.012	0.007	99.61
$\lambda_{\theta,\tau}$	-0.199	0.248	1.025	0.559	30.87	-0.081	0.212	0.902	0.471	40.49
$\lambda_{\theta,1/2}$	-0.163	0.214	0.788	0.387	41.65	-0.047	0.231	0.731	0.366	55.34
$100 \times \gamma_{\theta,\tau}$	-0.375	-0.194	-0.013	0.143	77.57	-0.242	-0.143	-0.061	0.076	88.93
$100 \times \gamma_{\theta,1/2}$	-0.300	-0.180	-0.045	0.099	86.21	-0.186	-0.115	-0.066	0.059	92.14

The table reports the summary statistics of the QL-CoVaR parameters estimated for the N financial companies included in our dataset. We estimated the conditional quantiles for two quantile levels of θ and $h = 0.15$. From left to right, we report the following descriptive statistics of the coefficients: the 5-th percentile (5P), the median (MED), the 95-th percentile (95P), the interquartile range (IQR) and the percentage of times, out of N, in which they are statistically significant at the 5% confidence level (PS).

described above, we estimate the QL-CoVaR parameters. The results are reported in Table 2.

As expected, we can see from Table 1 that positive returns of the individual companies have a positive impact on the VaR of the financial system, as $\lambda^{(i)}_{\theta}$ takes, on average, positive values. In contrast, Table 2 reports the statistics of the QL-CoVaR coefficients, where we condition the estimates to the distress and to the median state of a single financial company. As before, the average impact exerted by the companies to both $QL\text{-}CoVaR^{(i)}_{\tau}$ and $QL\text{-}CoVaR^{(i)}_{1/2}$ is positive, but greater with respect to the standard CoVaR (the medians of both $\widehat{\lambda}^{(i)}_{\theta,\tau}$ and $\widehat{\lambda}^{(i)}_{\theta,0.5}$ are greater than the median of $\widehat{\lambda}^{(i)}_{\theta}$). Therefore, the relationships between the system and the companies become stronger by focusing on particular regions of the $x_{i,t}$ support, i.e. when $x_{i,t}$ is in a neighbourhood of a distress state.

On average, we observe larger values for $\widehat{\lambda}^{(i)}_{\theta,\tau}$ at $\theta = 0.01$ than at $\theta = 0.05$, whereas the opposite holds for $\widehat{\lambda}^{(i)}_{\theta,0.5}$. $\widehat{\lambda}^{(i)}_{\theta,\tau}$ measures the relation between $x_{i,t}$ and y_t, when the companies and the system simultaneously lie in the left tail of their distributions. The fact that $\widehat{\lambda}^{(i)}_{\theta,\tau}$ increases as θ and τ simultaneously decrease means that the co-movements between the financial system and the companies become stronger when moving leftwards along the left tails of their distributions. Consequently, the risk of contagion increases by accentuating the distress degree in the connections between the financial system and the companies. For both CoVaR and QL-CoVaR, the percentage of times in which the coefficient measuring the impact of the individual companies (λ) is statistically significant at the 5% level is greater at $\theta = \tau = 0.05$ than at $\theta = \tau = 0.01$.

References

1. Adrian, T., Brunnermeier, K. (2016). CoVaR. *American Economic Review*, 106, 1705–1741.
2. Bondell, H.D., Reich, B.J., Wang, H. (2010). Non-crossing quantile regression curve estimation. *Biometrika*, 97, 825–838.
3. Efron, B. (1979). Bootstrap methods: Another look at the jackknife. *The Annals of Statistics*, 7, 1–26.
4. Koenker, R., Bassett, G. (1978). Regression quantiles. *Econometrica*, 46, 33–50.
5. Sim, N., Zhou, H. (2015). Oil prices, US stock return, and the dependence between their quantiles. *Journal of Banking & Finance*, 55, 1–8.