Probabilistic properties of Self Exciting Threshold Autoregressive processes

Proprietà probabilistiche dei processi autoregressivi a soglia

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Abstract In the present paper we focus the attention on the ergodicity (and stationarity) of the Self Exciting Autoregressive (SETAR) process. In more detail we review and compare the results given in the literature highlighting the main theoretical issues. Starting from these contributions, we debate, taking advantage of case studies, on the opportunity to further investigate on the statistical properties of the SETAR models and in particular on the possibility to weaken its stationarity conditions.

Abstract *Nel presente lavoro poniamo l'attenzione sulla ergodicità (e stazionarietà) dei processi autoregressivi a soglia. Più in dettaglio, proponiamo una rassegna e confrontiamo i principali contributi della letteratura dando particolare enfasi ai risultati teorici proposti. Tenuto conto di tali risultati discutiamo, avvalendoci di casi studio, sulla opportunità di esaminare ulteriormente le proprietà statistiche dei modelli autoregressivi a soglia e in particolare sulla opportunità di indebolire le condizioni di stazionarietà .*

Key words: ergodicity, threshold models, nonlinearity

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1 Introduction

In time series analysis the interest toward nonlinear models has received increasing attention. Since the seminal book of Tong (1990) a growing literature is appeared on this topic. A common factor in most of these contributions is the awareness that some results, largely investigated in linear time series domain, cannot be extended to models where the generating process is characterized by nonlinear dynamics (as widely agreed in the literature, we qualify *linear* those models that belong to the ARMA class (Box and Jenkins, 1976)).
In this context the threshold autoregressive (TAR) models (Tong, 1978) are a clear example of nonlinear stochastic structure where, instead of the local linearity of the process, most of the results of the linear time series models cannot be used.
To give evidence that TAR models are direct generalizations of the linear autoregressive structures, consider a stochastic process X_t , it is said to be a threshold autoregressive process if:

$$X_t = \sum_{k=1}^{\ell} \left(\phi_0^{(k)} + \sum_{i=1}^{p} \phi_i^{(k)} X_{t-i} + \varepsilon_t \right) \mathbb{I}(X_{t-d} \in \mathscr{R}_k) \tag{1}$$

where $\{\varepsilon_t\}$ is a sequence of independent and identically distributed (i.i.d.) random variables, $\mathbb{I}(\cdot)$ is an indicator function, ℓ is the number of regimes, p is the order of the autoregressive regimes, X_{t-d} is the threshold variable, d is the threshold delay, $\mathscr{R}_k = (r_{k-1}, r_k]$ is a subset of the real line such that $\mathscr{R} = \bigcup_{k=1}^{\ell} \mathscr{R}_k$ with $-\infty = r_0 < r_1 < \ldots < r_{k-1} < r_k = \infty$.

Starting from the definition of model (1), in the following we focus the attention on the ergodicity of the threshold autoregressive model. This topic has been differently faced in the literature, with varying degree of generality. In more detail: in Sect. 2 we first review the results on to the ergodicity of TAR models, which are discussed and compared; in Sect. 3 we debate on these results even showing, through case studies, that they can be weaken.

2 Ergodicity of TAR models

The ergodicity of the TAR model (1) has been differently examined. The preliminary results are given in Tong and Lim (1980) that first face the ergodicity of the SETAR($\ell; p$) model: they state that a sufficient condition is that the maximum eigenvalue of ${\Phi^{(k)}}^{\top} \times \Phi^{(k)}$ is less than one, with $\Phi^{(k)} = \begin{bmatrix} \phi_1^{(k)} & \phi_2^{(k)} & \cdots & \phi_p^{(k)} \\ & \mathbf{I_{p-1}} & & \mathbf{0} \end{bmatrix}$, $\mathbf{I}_{p-1}$ an identity matrix of order $p-1$, $\mathbf{0}$ a null vector, $\mathbf{A}^{\top}$ is the transpose of the matrix $\mathbf{A}$ and $k = 1, \ldots, \ell$. It implies that when $p = 1$ each regime needs to have $|\phi_1^{(k)}| < 1$ that, in other words, means that all regimes need to have roots outside the unit circle.
The Tong and Lim (1980) results have been revised in a short while in Petruccelli e

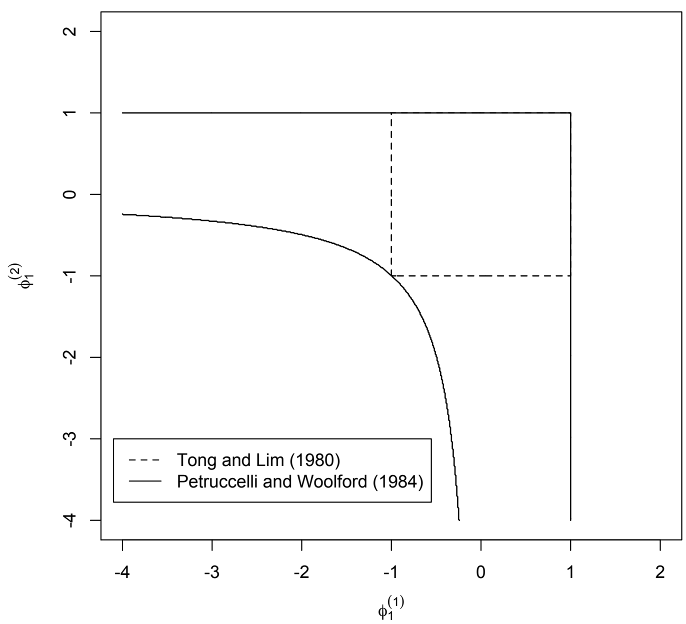

Fig. 1 Comparison of the ergodic regions, defined in Tong and Lim (1980) ans Petruccelli and Woolford (1984), of the SETAR (2;1) model (2)

Woolford (1984) that give weaker conditions. In particular they focus the attention on the SETAR(2;1) model, with threshold delay $d = 1$:

$$X_t = \left(\phi_1^{(1)} X_{t-1}\right) I_{t-1} + \left(\phi_1^{(2)} X_{t-1}\right)(1 - I_{t-1}) + \varepsilon_t, \tag{2}$$

with $I_{t-1} = 1$ if $X_{t-1} > r$ and $I_{t-1} = 0$ otherwise.
They state that the process (2) is ergodic if and only if

$$\phi_1^{(1)} < 1, \qquad \phi_1^{(2)} < 1, \qquad \phi_1^{(1)}\phi_1^{(2)} < 1. \tag{3}$$

The conditions (3) clearly expand the ergodic region of Tong and Lim (1980), as empirically shown in Fig. 1, and for the first time they state that the SETAR process can be ergodic even in presence of regions with roots inside the unit circle.

The conditions (3) have been generalized in Chan *et al.* (1985) for at least two main aspects: first they provide necessary and sufficient conditions for the ergodicity of the SETAR(ℓ;1) model; second, they consider autoregressive regimes where the intercepts can be different from zero. In particular Chan *et al.* (1985) state that the SETAR model (1) with $p = 1$ and $d = 1$ is ergodic if and only if:

$$\phi_1^{(1)} < 1, \quad \phi_1^{(k)} < 1, \quad \phi_1^{(1)}\phi_1^{(k)} < 1, \tag{4a}$$

$$\phi_1^{(1)} = 1, \quad \phi_1^{(k)} < 1, \quad \phi_0^{(1)} > 0, \tag{4b}$$

$$\phi_1^{(1)} < 1, \quad \phi_1^{(k)} = 1, \quad \phi_0^{(k)} < 0, \tag{4c}$$

$$\phi_1^{(1)} = 1, \quad \phi_1^{(k)} = 1, \quad \phi_0^{(k)} < 0 < \phi_0^{(1)}, \tag{4d}$$

$$\phi_1^{(1)}\phi_1^{(k)} = 1, \quad \phi_1^{(1)} < 0, \quad \phi_0^{(k)} + \phi_1^{(k)}\phi_0^{(1)} > 0, \tag{4e}$$

It can be noted that the intercept strongly impacts on the ergodicity of the process and so, as expected, on its dynamic structure. Chen and Tsay (1991) further note that another parameter that strongly affects the ergodic region is the threshold delay d (that in the previous contributions is always fixed to be one).
Given the SETAR(2;1) process with null intercepts, a necessary and sufficient condition for its geometrical ergodicity is:

$$\phi_1^{(1)} < 1, \quad \phi_1^{(2)} < 1, \quad \phi_1^{(1)}\phi_1^{(2)} < 1, \quad {\phi_1^{(1)}}^{s(d)}{\phi_1^{(2)}}^{t(d)} < 1, \quad {\phi_1^{(1)}}^{t(d)}{\phi_1^{(2)}}^{s(d)} < 1 \tag{5}$$

that when $d = 1$ correspond to those given in (3), whereas when $d > 1$, $s(d)$ and $t(d)$ are properly defined (Chen and Tsay, 1991, p. 615).
In Fig. 2 the ergodic region of Petruccelli and Woolford (1984) is compared to the regions defined in (5), when $d = 2$ (and so, following Chen and Tsay (1991), $s(d) = 1$ and $t(d) = 2$), $d = 3$ (with $s(d) = 3$ and $t(d) = 4$) and $d = 4$ (with $s(d) = 7$ and $t(d) = 8$).
It can be noted that a threshold delay, different from one, shrinks the ergodic region, that still remains unbounded.

The results presented until now focus the attention on models with autoregressive regimes of order $p = 1$. When $p > 1$ the conditions become more restrictive: it is shown in Chan and Tong (1985, Lemma 3.1) that a sufficient condition for the geometric ergodicity of model (1) with $d \leq p$ is:

$$\max_k \sum_{i=1}^{p} |\phi_i^{(k)}| < 1, \qquad k = 1, \ldots, \ell. \tag{6}$$

It can be noted that when $d = 1$ and $p = 1$ all regimes need to have roots outside the unit circle, so making the ergodic region more narrow than that given in Petruccelli and Woolford (1984).
Ling (1999) further investigate on the geometric ergodicity of the threshold model (1) and proposes (Ling, 1999, Theorem 6.4) the following sufficient condition:

$$\sum_{i=1}^{p} \max_k |\phi_i^{(k)}| < 1, \qquad k = 1, \ldots, \ell, \tag{7}$$

which defines a subset of the ergodic region given in (6) using heteroscedastic structures..

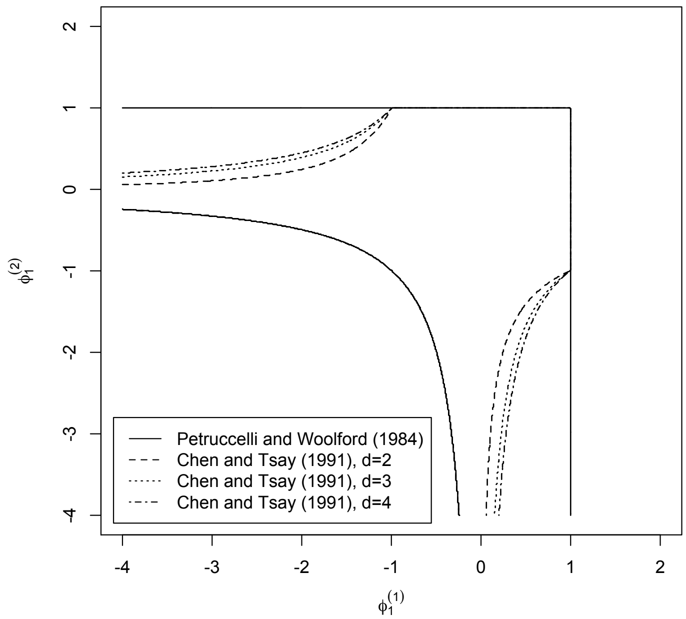

Fig. 2 Comparison of the ergodic regions, defined in Petruccelli and Woolford (1984) and Chen and Tsay (1991), of the SETAR (2;1) model for different values of the threshold delay d.

It is clearly shown in Fig. 3 where the conditions (6) and (7) are compared in presence of a SETAR(2;2) model. In particular we have considered two data generating process where the parameters of the second regimes have been fixed to $\phi_1^{(2)} = -0.4$ and $\phi_2^{(2)} = 0.1$ in frame (a) and to $\phi_1^{(2)} = -0.8$ and $\phi_2^{(2)} = 0.1$ in frame (b). As expected the growing of the absolute value of $\phi_1^{(2)}$ shrinks the Ling (1999) region that in both cases is included in the Chan and Tong (1985) region, as said before.

More recently Cline (2007) deals with the investigation of the ergodicity of threshold autoregressive models and propose the use of a procedure, called *piggyback*, whose technical details are given in Cline and Pu (2004). In practice the piggyback procedure allows to define the Lyapunov exponent in terms of an auxiliary process which is simpler that the threshold autoregressive one. The subsequent computation of the exponent is obtained using numerical evaluations which seem to be particularly efficient from the computational point of view. Moreover, the main problem that is found in Cline (2007) is that after computing the Lyapunov exponent, the ergodic region is obtained through a simulation study and, differently from the previous contributions, any condition is given on the model parameters. It strongly limits its applicability and the comparison to other results.

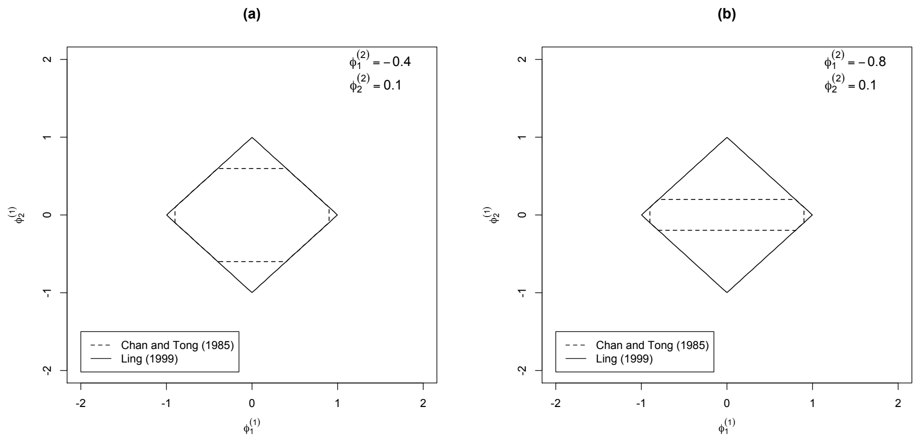

Fig. 3 Comparison of the Chan and Tong (1985) and Ling (1999) ergodic region for a SETAR(2;2) model with fixed valued of $\phi_1^{(2)}$ and $\phi_2^{(2)}$.

Note that all issues reviewed until now are obtained taking advantage of the Markovian nature of the SETAR process (as largely discussed in Tong (1990)). Knowing that the geometric ergodicity of Markov processes and given the initial stationary distribution, the conditions given before can be used to discuss the stationarity of the SETAR($\ell;p$) process. In this domain Bec *et al.* (2004) establish sufficient conditions for the stationarity of the SETAR($\ell=3;p$) process: they are based not only on the autoregressive coefficients of the three regimes but even on the value of the threshold delay d and the autoregressive order p.
When $\ell=2$, $p=2$ and $d=1$ those conditions become:

$$||(\boldsymbol{\Phi}^{(1)})^2||_{M_1}<1, \quad ||(\boldsymbol{\Phi}^{(2)})^2||_{M_1}<1, \quad ||\boldsymbol{\Phi}^{(1)}\boldsymbol{\Phi}^{(2)}||_{M_1}<1 \tag{8}$$

where the norm $||\mathbf{A}||_{M_1}=||\mathbf{D}_t\mathbf{U}^\top\mathbf{A}\mathbf{U}\mathbf{D}_t^{-1}||_1$, with $\mathbf{U}$ the orthogonal matrix obtained from the Schur decomposition of $\mathbf{A}$, $\mathbf{D}_t=\text{diag}(t,t^2)$, given a value t, and the norm $||\mathbf{Q}||_1=\max_{j\in\{1,2\}}\sum_{i=1}^2|q_{ij}|$.

The conditions (8) are not obvious (and are related to the choice of t) but they can be used to define more simple sufficient conditions that can be easily compared to the results presented before. In particular, knowing that the dominant eigenvalue $\rho(\mathbf{A})\le||\mathbf{A}||_{M_1}$ (Horn and Johnson, 2013, Theorem 5.6.9) and that $\rho(\mathbf{A}^n)=\rho(\mathbf{A})^n$, for $n=1,2,\dots,N$, it implies that the SETAR(2;2) process, with $d=1$, is stationary if:

$$\rho(\boldsymbol{\Phi}^{(1)})<1, \quad \rho(\boldsymbol{\Phi}^{(2)})<1, \quad \rho(\boldsymbol{\Phi}^{(1)}\boldsymbol{\Phi}^{(2)})<1. \tag{9}$$

In other words, from Theorem 1 of Bec *et al.* (2004) and using some widely known properties of the matrix norm, the stationarity of the SETAR(2;2) model can

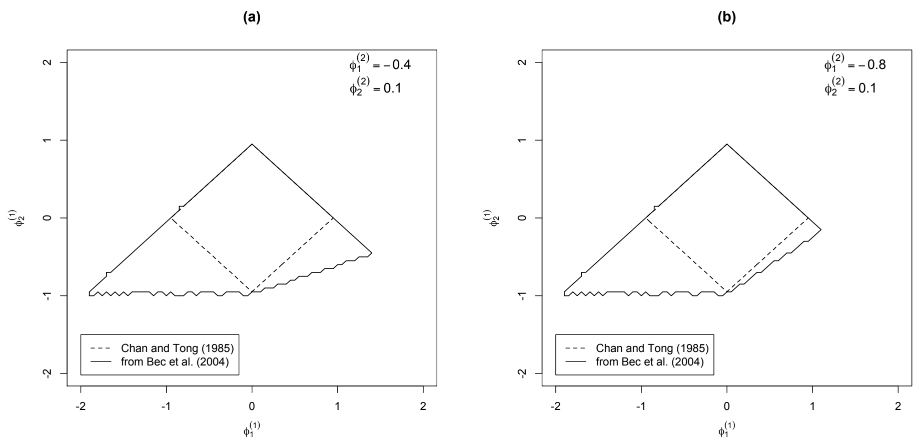

Fig. 4 Comparison of the stationarity regions, obtained from Chan and Tong (1985) and the conditions (9), for a SETAR(2;2) model with fixed valued of $\phi_1^{(2)}$ and $\phi_2^{(2)}$.

be achieved if the two regimes have roots outside the unit circle and if $\rho(\Phi^{(1)}\Phi^{(2)}) < 1$.
It is easy to verify that when $p = 1$ the conditions (9) are included in those given in (3), whereas if they are compared to the results of Chan and Tong (1985), it can be noted the (9) allows to obtain a wider stationarity region over the parametric space. It can be observed in Fig.4 where the two regions are represented for $\phi_1^{(1)} \in [-2;2]$, $\phi_2^{(1)} \in [-2;2]$ and $\phi_1^{(2)} = -0.4$, $\phi_2^{(2)} = 0.1$ in frame (a) and $\phi_1^{(2)} = -0.8$, $\phi_2^{(2)} = 0.1$ in frame (b).

A common factor that gathers the results (6), (7) and (9) is that when $\ell = 2$, $p = 1$ and $d = 1$ the stationarity of the SETAR(2;1) process requires, among the others, both regimes with roots outside the unit circle. Further, when $p = 2$, they become more restrictive. In the next section we show, through case studies, that these conditions could be weakened.

3 Case studies

To discuss the results presented in Sect. 2., consider the following examples.

Example 1. Let $X_t \sim$SETAR(2;2) with $d = 1$ and $r = 0$:

$$X_t = \left(\phi_1^{(1)}X_{t-1} + \phi_2^{(1)}X_{t-2}\right)I_{t-1} + \left(\phi_1^{(2)}X_{t-1} + \phi_2^{(2)}X_{t-2}\right)(1 - I_{t-1}) + \varepsilon_t \qquad (10)$$

where $\{\varepsilon_t\}$ is a sequence of i.i.d. random variables with $E[\varepsilon_t] = 0$ and $E[\varepsilon_t^2] = 1$.

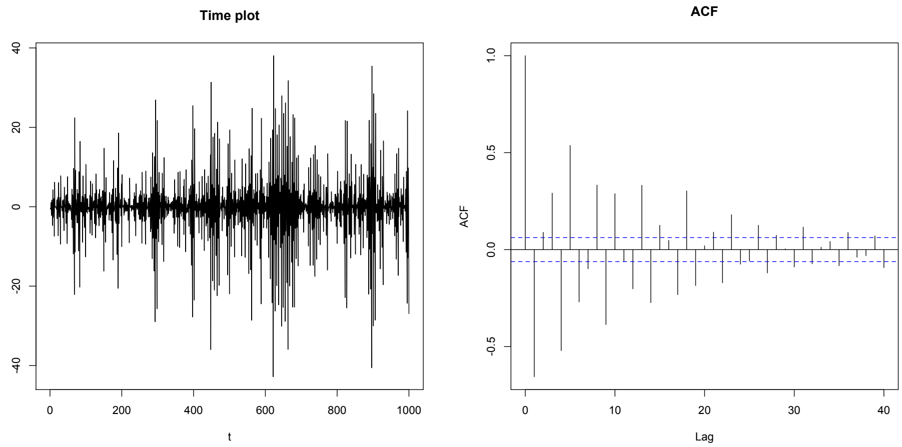

Fig. 5 Time plot and ACF of artificial data simulated in Example 2

We have simulated from model (10) a time series of length $n = 1000$ (with burn-in 500) with the following autoregressive parameters: $\phi_1^{(1)} = -2.6$, $\phi_2^{(1)} = -2.1$, $\phi_1^{(2)} = -0.9$ and $\phi_2^{(2)} = -0.1$, where $\rho(\Phi^{(1)}) = 1.4491$ and $\rho(\Phi^{(2)}) = 0.7701$. In this case the first regime has roots inside the unit circle whereas the second regime has both roots outside the unit circle. As can be observed from Fig. 5 , in this case the process seems to be stationary despite the inequalities (6), (7) and (9) do not hold.

Example 2. Consider now this second example where the artificial data are generated from model (10) with $\phi_1^{(1)} = -2.6$, $\phi_2^{(1)} = -2.1$, $\phi_1^{(2)} = -1.1$ and $\phi_2^{(2)} = -0.1$, where $\rho(\Phi^{(2)}) = 1$.
From Fig. 6 it can be observed that the nonstationarity of the two regimes does not affect the stationarity of the SETAR model. Further the selected coefficients are definitely far from the stationary region defined in Chan and Tong (1985) and Ling (1999).

Starting from these results, that clearly highlight the opportunity to obtain more wide stationarity regions that those given in the previous pages, Giordano *et al.* (2016) have proposed new issues. In particular they have defined a new stationary region over the parametric space, where the results (6) and (7) are a strict subset.
They further summarize the main differences between the conditions for the stationairty of the SETAR(2;1) and the SETAR(2;2) models that can be sketched in this way:

SETAR(2;1) case

(a) both regimes have roots outside the unit circle $\Rightarrow$ *SETAR(2;1)* **is** *stationary*
(b) both regimes have roots inside the unit circle $\Rightarrow$ *SETAR(2;1)* **is not** *stationary*

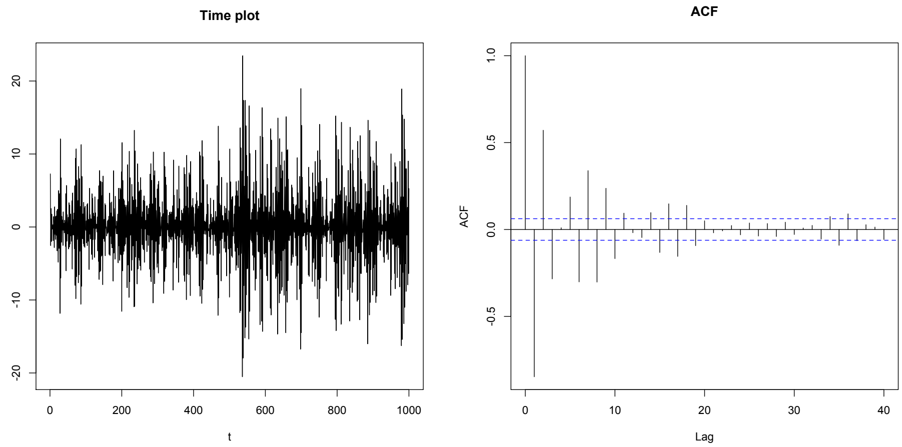

Fig. 6 Time plot and ACF of artificial data simulated in Example 3

(c) one of the two regimes has roots inside the unit circle ⇒ *SETAR(2;1)* **could be** *stationary*

SETAR(2;2) case

(a),(b),(c) ⇒ *SETAR(2;2)* **could be** *stationary*

This scheme shortly summarizes how complex can be the study of the stationarity of SETAR models when $p > 1$ and the reason because it can be still considered an open problem for SETAR($\ell; p$) structures.

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