

Conditional Value-at-Risk: a comparison between quantile regression and copula functions

Conditional Value-at-Risk: un confronto tra la regressione quantile e le funzioni copula

Giovanni De Luca and Giorgia Rivieccio

Abstract The popular Value-at-Risk of an institution provides a measure of its own risk. However, in many cases it is of interest to know the measure of the contribution of an institution to the systemic risk, based on the Conditional Value-at-Risk. In this paper we compare the estimation of such measure according to two different methods. The former is based on the quantile regression, the latter on copula functions. In both cases, the heteroskedasticity of the time series is explicitly taken into account through a GARCH structure. Moreover, the comparison is also made across different distributional assumptions.

Abstract *Il Value-at-Risk è un'importante misura di rischio di un'istituzione. Tuttavia, in molti casi, è di maggiore interesse la misura del contributo di un'istituzione al rischio sistemico. Tale misura è basata sul Conditional Value-at-Risk. In questo lavoro si confrontano due metodi per la stima del contributo, il primo basato sulla regressione quantile, il secondo sulle funzioni copula. In entrambi i casi la struttura eteroschedastica è inclusa attraverso una struttura GARCH. Il confronto è svolto anche considerando diverse ipotesi distributive.*

Key words: systemic risk, quantile regression, copula functions.

1 Introduction

Literature on systemic risk has been recently enriched by the introduction of a new measure, the Conditional Value-at-Risk (*CoVaR*), measuring the effect of an individual financial institution to the systemic risk. *CoVaR* has been proposed in Adrian and Brunnermeier (2011). In its general formulation, it is defined as the Value-at-

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Risk (VaR) of the financial system given that an institution is under distress. The importance of this measure is given by the possibility of quantifying the contribution of an institution to the financial system distress.

2 Methodology

The Value-at-Risk at level q of the financial system is defined as the q -th quantile of the distribution of the returns Y_f of an index representing the financial system, i.e.

$$Pr(Y_f \leq VaR_q^f) = q.$$

We denote by $CoVaR_q^{f|Y_i=VaR_q^i}$ the VaR of the index conditional on financial distress of institution i , that is conditional on $Y_i = VaR_q^i$, where Y_i is the return of institution i . As a result, $CoVaR_q^{f|i}$ is implicitly defined by the q -th quantile of the conditional probability distribution $Y_f|Y_i$, that is

$$P(Y_f \leq CoVaR_q^{f|i} | Y_i = VaR_q^i) = q$$

This is the original definition that will be considered in this paper.¹

The $CoVaR$ can be interpreted as the estimate of the effect on the system of a critical situation of institution i . Moreover, it is defined the so-called $\Delta CoVaR_q^{f|i}$, given by

$$\Delta CoVaR_q^{f|i} = CoVaR_q^{f|Y_i=VaR_q^i} - CoVaR_q^{f|Y_i=Median^i}$$

where $CoVaR_q^{f|X_i=Median^i}$ is the q -th quantile of Y_f conditional on a normal situation of institution i , that is the return of institution i is equal to its median. The $\Delta CoVaR_q^{f|i}$ is the increase of $CoVaR_q^{f|i}$ when the institution i goes into distress.

The estimation of the $CoVaR_q^{f|i}$ is usually based on the linear quantile regression technique (Koenker and Bassett, 1978). In this case, the q -th quantile of Y_f , denoted as Y_q^f , is estimated as function of Y_i

$$\hat{Y}_q^f = \hat{\alpha}_q + \hat{\beta}_q Y_i$$

So, the estimate of $CoVaR_q^{f|i}$ is given by

$$\widehat{CoVaR}_q^{f|Y_i=VaR_q^i} = \hat{\alpha}_q + \hat{\beta}_q VaR_q^i$$

while the estimate of $CoVaR_q^{f|Y_i=Median}$ is given by

¹ Girardi and Ergun (2013) have modified the original definition of $CoVaR$, enlarging the conditional event (the institution i is at maximum at VaR_q^i , that is $Y_i \leq VaR_q^i$).

$$\widehat{CoVaR}_q^{f|Y_i=Median^i} = \hat{\alpha}_q + \hat{\beta}_q Median^i.$$

Finally,

$$\Delta \widehat{CoVaR}_q^{f|i} = \hat{\beta}_q (VaR_q^i - Median^i).$$

This procedure does not take into account the heteroskedasticity that typically characterizes financial returns. In this work we modify the quantile regression to take into account this feature of returns. The modified quantile regression approach is based on the following steps:

- (a) estimate of a GARCH-type model for returns of both the index of the financial system and institution i and derivation of the standardized residuals η_f and η_i ;
- (b) estimate of the parameters α and β using a quantile regression such that the q -th quantile of η_f is given by

$$\hat{\eta}_q^f = \hat{\alpha}_q + \hat{\beta}_q \eta^i$$

- (c) transformation of the estimated conditional quantiles $\hat{\eta}_q^f$ to obtain $\Delta CoVaR_q^{f|i}$.

An alternative approach is the estimate of $\Delta CoVaR_q^{f|i}$ using the Copula-GARCH approach (Jondeau and Rockinger, 2006). The steps of this procedure are:

- (a) estimate of a GARCH-type model for returns of both the index of the financial system and institution i and derivation of the standardized residuals η_f and η_i ;
- (b) selection of a copula function describing the bivariate relationship between η_f and η_i , $C(F_{\eta_f}, F_{\eta_i})$, where F_{η_f} and F_{η_i} are the distribution functions of the variables η_f and η_i , respectively;
- (c) derivation of the conditional copula, $C(F_{\eta_f}|F_{\eta_i})$;
- (d) transformation of the estimated quantiles $\hat{\eta}_q^f$ from the conditional copula to obtain $\Delta CoVaR_q^{f|i}$.

Finally, the two procedures are compared through the comparison of the estimates of $\Delta CoVaR_q^{f|i}$.

3 Application to data

We have considered the daily log-returns of eight assets: A2A, BPER, Enel, FCA, Generali, Intesa San Paolo, STM, Unicredit. Three of them belong to the banking sector (BPER, Intesa San Paolo and Unicredit). The returns have been observed in the period from February 27th, 2013 to January 30th, 2018 (in total we have 1249 observations). The financial system is represented by the FTSEMIB index.

In the quantile regression approach, the estimates have been carried out considering different distributional assumptions in the GARCH specification (in particular we have considered three conditional distribution: Normal, Student's t and Skew Student's t). The same distributional assumptions have been considered in the Copula-GARCH approach, while the copula function is the Student's t copula, characterized by lower and upper tail dependence.

Figure 1 reports the scatterplots of average $\Delta CoVaR_{0,05}^{f|i}$ against the empirical q -th quantile for the eight assets. The top row contains the scatterplots with $\Delta CoVaR_{0,05}^{f|i}$ computed using the quantile regression approach, in the bottom the scatterplots consider the $\Delta CoVaR_{0,05}^{f|i}$ estimated based on the copula approach. The columns identify the distributional assumptions: Normal (left), Student's t (middle) and Skew Student's t (right). To facilitate the comparison, we have kept the same scale on the y-axis.

The most relevant findings are the following:

1. The quantile regression approach tend to group the assets in two cluster, regardless the distribution of the returns. Interestingly, the two assets with the lowest $\Delta CoVaR_{0,05}^{f|i}$ are two banking institutions (Intesa SanPaolo and Unicredit, that is the biggest Italian banks) shown in the box.
2. According to the copula approach, no clear clustering of the eight assets seems to be plausible. On the other hand, a common feature is evident: Intesa SanPaolo and Unicredit are still the assets with the lowest $\Delta CoVaR_{0,05}^{f|i}$ (still shown inside the box).
3. The values of $\Delta CoVaR_{0,05}^{f|i}$ provided by the copula approach are generally smaller. The details can be found in Table 1 reporting the percentage of cases the $\Delta CoVaR_{0,05}^{f|i}$ obtained by the procedure in row is smaller than the corresponding measure obtained using the procedure in column. We can conclude that the use of the Student's t copula function which admits tail dependence ensures a more flexible description of the association in the tails of the distributions allowing smaller quantiles.

Table 1 Percentages of success (smaller value) of $\Delta CoVaR_{0,05}^{f|i}$ provided by the copula procedure (marginal distributions are in brackets).

	QR (Gaussian)	QR (Student's t)	QR (Skew Student's t)
Copula (Gaussian)	75%	75%	75%
Copula (Student's t)	100%	100%	100%
Copula (Skew Student's t)	100%	100%	100%

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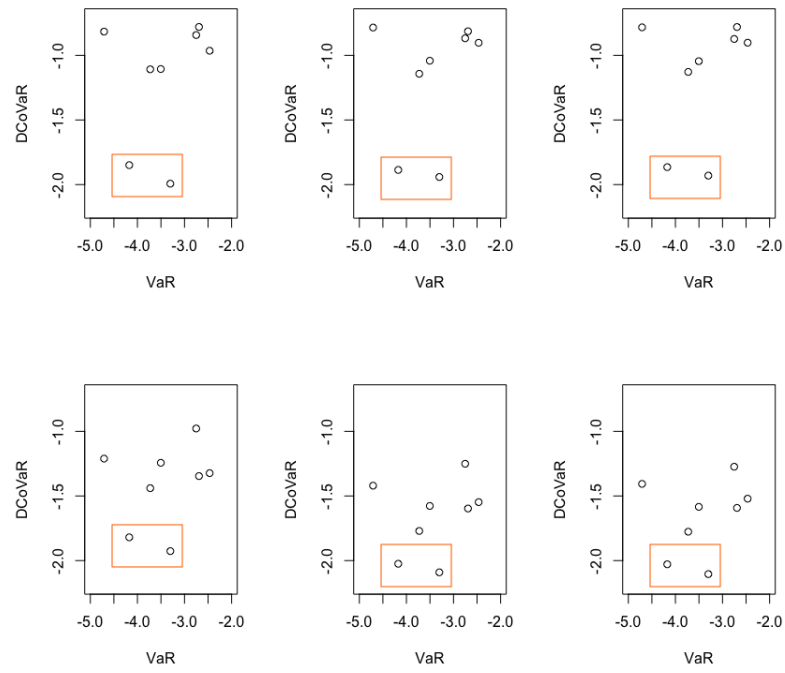


Fig. 1 Scatter plot of VaR- Δ CoVaR for the eight assets ($q = 0.05$). Two methods have been considered: quantile regression (top) and copula functions (below). Moreover, different distributional assumptions have been considered: Normal (left), Student's t (middle) and Skew Student's t (right).