Complex Contingency Tables and Partitioning of Three-way Association Indices for Assessing Justice Court Workload

Tabelle di Contingenza Complesse e Partizioni di Indici di Associazione per la Valutazione del Carico di Lavoro nei Tribunali Giudiziari

Rosaria Lombardo, Yoshio Takane and Eric J Beh

Abstract A comprehensive study is conducted on the partition of two common indices for three-way contingency tables under several representative hypotheses about the expected frequencies (hypothesized probabilities). Specifically, the partition of the classical (symmetrical) three-way Pearson index and of the asymmetrical three-way Marcotorchino index are considered under a general *Scenario 0* from which known different scenarios are derived: 1) where the hypothesized probabilities are homogeneous among the categories [12], and 2) when the hypothesized probabilities are estimated from the data [7, 6].
Abstract *In questo lavoro si presenta uno studio completo sulla partizione di due indici di associazione per le tabelle di contingenza a tre vie, in base a diverse ipotesi sulle frequenze attese (probabilitá ipotizzate). Nello specifico, si propone la partizione dell'indice di Pearson (simmetrico) e dell'indice di Marcotorchino a tre vie (asimmetrico) in uno Scenario 0 da cui derivano note partizioni: 1) dove le probabilitá ipotizzate sono uniformi [12] e 2) quando le probabilitá ipotizzate sono stimate dai dati [7, 6].*

Key words: Three-way association indices, Hypothesized probabilities, Observed frequencies, Chi-Squared distribution

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R. Lombardo
Economics Department, University of Campania, via Gran Priorarato di Malta, Capua (CE), Italy, Tel.: +390810601382, Fax: +390823622984, e-mail: rosaria.lombardo@unina2.it

Y. Takane
University of Victoria, 5173 Del Monte Ave, Victoria BC, V8Y 1X3, Canada e-mail: yoshio.takane@mcgill.ca

E.J. Beh
School of Mathematical and Physical Sciences, University of Newcastle, Callaghan, 2308, NSW, Australia e-mail: eric.beh@newcastle.edu.au

1 Introduction

Measuring and assessing the association among categorical variables can be undertaken by partitioning a multi-way association index into bivariate, trivariate and higher order terms. The partitions presented in this paper are based on the work by Lancaster (1951) which considered an ANOVA-like partitions of Pearson's chi-squared statistic implemented for the analysis of a $2 \times 2 \times 2$ table. Here, we focus on Pearson's statistic [10] and on Marcotorchino's index [15] for studying the symmetrical and asymmetrical association among three categorical variables, respectively. In this paper we show that, under complete independence of the three categorical variables, a general scenario - called *Scenario 0* - is defined from which different known partitions of Pearson's statistic and Marcotorchino's index can be derived as special cases. Examples of these known partitions include those of [7, 6], [14], [3, 4] and [12]. In *Scenario 0*, the probabilities are prescribed by the analyst instead of being estimated by using the margins of the empirical distribution that underly the data. The reason is that *a priori* knowledge of phenomena can suggest differently; see [1, 12].

The paper is organised in the following manner. After introducing the notation, a general partition of a three-way array in $\Re^{I\times J\times K}$ is discussed in Section 2. The key scenarios of model dependence are described in Section 3. Section 4 presents the general partition for Pearson's statistic. A practical demonstration of this partition under *Scenario 0* is given in Section 5. Some concluding remarks are made in Section 6.

2 Partitioning three-way association indices

Consider a general three-way contingency table $\underline{\mathbf{X}} = \left(x_{ijk}\right) \in \Re^{I\times J\times K}$ (for $i = 1,\ldots,I$, $j = 1,\ldots,J$, and $k = 1,\ldots,K$) that summarises the cross-classification of n individuals/units according to I row, J column and K tube categories. These sets of categories form the row, column and tube variables x_I, x_J, and x_K, respectively. Let $\underline{\hat{\mathbf{P}}} = (\hat{p}_{ijk})$ be the array of the observed joint relative frequencies (of dimension $I \times J \times K$), so that $\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} x_{ijk}/n = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} \hat{p}_{ijk} = 1$.

Let $\underline{\mathbf{P}} = (p_{ijk})$ be the array of the joint probability values of the cells in the three-way tables. Define $\mathbf{p}_I\{= p_{i\bullet\bullet}\}$, $\mathbf{p}_J\{= p_{\bullet j\bullet}\}$, $\mathbf{p}_K\{= p_{\bullet\bullet k}\}$ to be the vectors of the marginal probabilities associated with the three variables x_I, x_J, and x_K, respectively.

In case complete three-way independence is hypothesized, it holds that $p_{ijk} = p_{i\bullet\bullet}p_{\bullet j\bullet}p_{\bullet\bullet k}$, where p_{ijk} is the $(i,\, j,\, k)$th joint probability and $p_{i\bullet\bullet}$, $p_{\bullet j\bullet}$ and $p_{\bullet\bullet k}$ are the marginal probabilities of the ith row category, the jth column category, and the kth tube category, respectively.

When studying the association among variables that are symmetrically associated, Pearson's chi-squared statistic is always appropriate. Instead, when variables are asymmetrically associated (for example, x_I may be considered as the response variable while x_J and x_K are treated as the predictor variables) then Marcotorchino's index [15, 14, 4] may be considered more suitable.

Let $\underline{\boldsymbol{\Pi}}$ be a three-way array whose general element is $\pi_{ijk} = \left(\frac{\hat{p}_{ijk}}{p_{i\bullet\bullet}p_{\bullet j\bullet}p_{\bullet\bullet k}} - 1\right)$ and define $\underline{\boldsymbol{\Pi}}_M = \left(\frac{\hat{p}_{ijk}}{p_{\bullet j\bullet}p_{\bullet\bullet k}} - p_{i\bullet\bullet}\right)$. Let $\underline{\mathbf{M}} = \mathbf{D_I} \otimes \mathbf{D_J} \otimes \mathbf{D_K}$ and $\underline{\mathbf{M}}_{tau} = \mathbf{I} \otimes \mathbf{D_J} \otimes \mathbf{D_K}$, be the metric related to the arrays $\underline{\boldsymbol{\Pi}}$ and $\underline{\boldsymbol{\Pi}}_M$, respetively. Given the traditional definition of inner products and quadratic norm in the space $\Re^{I\times J\times K}$ [16, p. 7],[6], observe that the quadratic norm of $\underline{\boldsymbol{\Pi}}$ with metric $\underline{\mathbf{M}}$ represents Pearson's mean square statistic and can be expressed as

$$\frac{\chi^2}{n} = \Phi^2 = \| \underline{\boldsymbol{\Pi}} \|_M^2 = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} p_{i\bullet\bullet}p_{\bullet j\bullet}p_{\bullet\bullet k}\pi_{ijk}^2\,, \tag{1}$$

which measures the overall discrepancy between a set of observed frequencies and the expected frequencies.

Similarly, the quadratic norm of $\underline{\boldsymbol{\Pi}}_M$ with metric $\underline{\mathbf{M}}_{tau}$ represents Marcotorchino's index numerator and can be expressed as

$$\tau_M^{num} = \| \underline{\boldsymbol{\Pi}} \|_{M_{tau}}^2 = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} p_{\bullet j\bullet}p_{\bullet\bullet k}(\pi_{ijk}^M)^2\,, \tag{2}$$

which measures the discrepancy between a set of conditional frequencies and the expected frequencies. The denominator of this index is considered a constant term, since it does not depend on the predictor variables, and is usually neglected. Marcotorchino's index, called τ_M, is a generalization of the τ index presented by [8] for studying the predictability issue in two-way contingency tables. For more information on this index, refer to [15, 14] and [4, p.461].

For partitioning purpose, examine the orthogonal projections of a general three-way array $\underline{\mathbf{X}}$ beloging to the space $\Re^{I\times J\times K}$ onto the subspaces $\Re^0$, $\Re^I$, $\Re^J$, $\Re^K$, $\Re^{JK}$, $\Re^{IJ}$, $\Re^{IK}$ and $\Re^{IJK}$. Then, according to the ANOVA-like decomposition of the elements of an array we get

$$x_{ijk} = a + b_i + c_j + d_k + e_{ij} + f_{ik} + g_{jk} + h_{ijk}\,, \tag{3}$$

so that there exists a fixed main term ($a = x_{\bullet\bullet\bullet}$), three univariate main terms ($b_i = x_{i\bullet\bullet} - x_{\bullet\bullet\bullet}$, $c_j = x_{\bullet j\bullet} - x_{\bullet\bullet\bullet}$ and $d_k = x_{\bullet\bullet k} - x_{\bullet\bullet\bullet}$), three bivariate terms ($e_{ij} = x_{ij\bullet} - x_{\bullet\bullet\bullet}$, $f_{ik} = x_{i\bullet k} - x_{\bullet\bullet\bullet}$ and $g_{jk} = x_{\bullet jk} - x_{\bullet\bullet\bullet}$) and a trivariate effect ($h_{ijk} = x_{ijk} - x_{ij\bullet} - x_{i\bullet k} - x_{\bullet jk} + x_{i\bullet\bullet} + x_{\bullet j\bullet} + x_{\bullet\bullet k} - x_{\bullet\bullet\bullet}$), defined onto the sub-spaces $\Re^0$, $\Re^I$, $\Re^J$, $\Re^K$, $\Re^{IJ}$, $\Re^{IK}$, $\Re^{JK}$ and $\Re^{IJK}$, respectively. The determination of the partition terms depend upon the definition of the array $\underline{\mathbf{X}}$ and on the metric related to each subspace, as it will be illustrated in Section 2.1

2.1 Index's partition term

In general, the uniqueness of each term of the partition in Equation 3, for example say b_i, is verified if the following two conditions are satisfied

1. each term belongs to the related space: $b_i \in \Re^I$
2. each term is orthogonal to the space vectors:

$$(\mathbf{x} - \mathbf{b}_M) \perp \Re^I \leftrightarrow < \mathbf{x} - \mathbf{b}, \beta >_M = 0 \qquad \forall \beta \in \Re^I$$

as consequence

$$< \mathbf{x}, \beta >_M = < \mathbf{b}, \beta >_M$$

By setting $\boldsymbol{x} = \boldsymbol{\pi}$, using the metric $\underline{\mathbf{M}}$ and expanding the inner product, we get

$$\begin{aligned}
\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} p_{i\bullet\bullet}p_{\bullet j\bullet}p_{\bullet\bullet k}\pi_{ijk}\beta_i &= \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} p_{i\bullet\bullet}p_{\bullet j\bullet}p_{\bullet\bullet k}b_i\beta_i \\
\sum_{j=1}^{J}\sum_{k=1}^{K} p_{\bullet j\bullet}p_{\bullet\bullet k}\pi_{ijk} &= b_i \\
\pi_{i\bullet\bullet} &= b_i \,.
\end{aligned}$$

So the projection of $\underline{\boldsymbol{\Pi}}$ onto the subspace $\Re^I$ is defined by

$$\begin{aligned}
b_i &= \pi_{i\bullet\bullet} - \pi_{\bullet\bullet\bullet} \\
&= \sum_{j=1}^{J}\sum_{k=1}^{K} p_{\bullet j\bullet}p_{\bullet\bullet k}\left(\frac{\hat{p}_{ijk}}{p_{i\bullet\bullet}p_{\bullet j\bullet}p_{\bullet\bullet k}} - 1\right) \\
&= \left(\frac{\hat{p}_{i\bullet\bullet}}{p_{i\bullet\bullet}} - 1\right)
\end{aligned}$$

The weighted quadratic norm of this term is equal to the first term on the right-hand side of Equation 4.
The orthogonal projections of $\underline{\boldsymbol{\Pi}}$ onto the remaining subspaces can be similarly defined.
Furthermore, changing the metric in $\underline{\mathbf{M}}_{tau} = \boldsymbol{I} \otimes D_J \otimes D_K$ and taking into account the array $\underline{\boldsymbol{\Pi}}_M$, we can get the orthogonal projections of Marcotorchino's index in a similar way.

3 The key scenarios of model dependence

Under the three-way model of complete independence, we present a new general scenario - referred to as *Scenario 0* - which can be applied to each variable of a three- or multi-way contingency table rather than to contingency tables as a whole (see Section 4). In *Scenario 0*, the marginal probabilities, $\mathbf{p}_I$, $\mathbf{p}_J$, and $\mathbf{p}_K$, can take on any prescribed values that satisfy the classical probability laws (e.g., $p_{i\bullet\bullet} \geq 0$, and $\sum_{i=1}^{I} p_{i\bullet\bullet} = 1$). We do not assume that the probabilities of $\mathbf{p}_I$, $\mathbf{p}_J$, and $\mathbf{p}_K$ are user-defined but are dictated by situational demand.

- *Scenario 1*. A special case of *Scenario 0* where it is hyphotesised the marginal homogeneity under independence such that $\mathbf{p}_I = \mathbf{1}_I/I, \mathbf{p}_J = \mathbf{1}_J/J$, and $\mathbf{p}_K = \mathbf{1}_K/K$, where $\mathbf{1}_I, \mathbf{1}_J$ and $\mathbf{1}_K$ are unitary vector of length I, J and K, respectively. In this case the degree of freedom, *df*, for the three-way chi-squared statistic is $IJK - 1$. This scenario is at the core of the chi-squared partition proposed by Loisel and Takane by using orthogonal transformations of variables [12].
- *Scenario 2*. A special case of *Scenario 0* in which the probabilities in $\mathbf{p}_I$, $\mathbf{p}_J$, and $\mathbf{p}_K$ are estimated from the data, so that $\mathbf{p}_I = \hat{\mathbf{p}}_I$, $\mathbf{p}_J = \hat{\mathbf{p}}_J$ and $\mathbf{p}_K = \hat{\mathbf{p}}_K$, are prescribed to be equal to the observed marginal proportions of the three categorical variables. In this scenario, the *df* for the three-way chi-squared statistic is $(IJK-1)-(I-1)-(J-1)-(K-1)$
- *Scenario 3*. A special case of *Scenario 0* in which the specification of the probabilities in $\mathbf{p}_I$, $\mathbf{p}_J$, and $\mathbf{p}_K$ can be a mix of both *Scenario 1* and *Scenario 2*.

For all scenarios, the partition terms of Pearson's mean square coefficient and Marcotorchino's index in the space $\Re^{I\times J\times K}$ are likely to be different. However, irrespective of which special case of *Scenario 0* is considered, these statistics and each term of their partition, can be tested (asymptotically) for statistical significance using a χ^2 distribution.

4 Partitioning Pearson's statistic

For a sake of brevity, here we illustrate only the partition of Φ^2 under *Scenario 0*. The row, column and tube marginal probabilities, $\mathbf{p}_I$, $\mathbf{p}_J$, and $\mathbf{p}_K$, respectively, can be *a priori* known or estimated from the data. This general partition is

$$\begin{aligned}
\Phi^2 &= \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} \frac{1}{p_{i\bullet\bullet}p_{\bullet j\bullet}p_{\bullet\bullet k}} (\hat{p}_{ijk} - p_{i\bullet\bullet}p_{\bullet j\bullet}p_{\bullet\bullet k})^2 \\
&= \sum_{i=1}^{I} \frac{1}{p_{i\bullet\bullet}} (\hat{p}_{i\bullet\bullet} - p_{i\bullet\bullet})^2 + \sum_{j=1}^{J} \frac{1}{p_{\bullet j\bullet}} (\hat{p}_{\bullet j\bullet} - p_{\bullet j\bullet})^2 + \sum_{k=1}^{K} \frac{1}{p_{\bullet\bullet k}} (\hat{p}_{\bullet\bullet k} - p_{\bullet\bullet k})^2 \\
&\quad + \sum_{i=1}^{I}\sum_{j=1}^{J} \frac{1}{p_{i\bullet\bullet}p_{\bullet j\bullet}} (\hat{p}_{ij\bullet} - \hat{p}_{i\bullet\bullet}p_{\bullet j\bullet} - \hat{p}_{\bullet j\bullet}p_{i\bullet\bullet} + p_{i\bullet\bullet}p_{\bullet j\bullet})^2 \\
&\quad + \sum_{i=1}^{I}\sum_{k=1}^{K} \frac{1}{p_{i\bullet\bullet}p_{\bullet\bullet k}} (\hat{p}_{i\bullet k} - \hat{p}_{i\bullet\bullet}p_{\bullet\bullet k} - \hat{p}_{\bullet\bullet k}p_{i\bullet\bullet} + p_{i\bullet\bullet}p_{\bullet\bullet k})^2 \\
&\quad + \sum_{j=1}^{J}\sum_{k=1}^{K} \frac{1}{p_{\bullet j\bullet}p_{\bullet\bullet k}} (\hat{p}_{\bullet jk} - \hat{p}_{\bullet j\bullet}p_{\bullet\bullet k} - \hat{p}_{\bullet\bullet k}p_{\bullet j\bullet} + p_{\bullet j\bullet}p_{\bullet\bullet k})^2 \\
&\quad + \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} \frac{1}{p_{i\bullet\bullet}p_{\bullet j\bullet}p_{\bullet\bullet k}} (\hat{p}_{ijk} - \hat{p}_{ij\bullet}p_{\bullet\bullet k} - \hat{p}_{i\bullet k}p_{\bullet j\bullet} - \hat{p}_{\bullet jk}p_{i\bullet\bullet} \\
&\qquad + \hat{p}_{i\bullet\bullet}p_{\bullet j\bullet}p_{\bullet\bullet k} + \hat{p}_{\bullet j\bullet}p_{i\bullet\bullet}p_{\bullet\bullet k} + \hat{p}_{\bullet\bullet k}p_{i\bullet\bullet}p_{\bullet j\bullet} - p_{i\bullet\bullet}p_{\bullet j\bullet}p_{\bullet\bullet k})^2 \\
&= \Phi_I^2 + \Phi_J^2 + \Phi_K^2 + \Phi_{IJ}^2 + \Phi_{IK}^2 + \Phi_{JK}^2 + \Phi_{IJK}^2. \qquad (4)
\end{aligned}$$

Here we can see that there are seven terms in the partition. The first term is the row main effect which, when multiplied by n and under the null hypothesis $p_{ijk} = p_{i\bullet\bullet}p_{\bullet j\bullet}p_{\bullet\bullet k}$, asymptotically follows a χ^2_{I-1} distribution, whose $1-\alpha$ percentile is $\chi^2_{\alpha,I-1}$ with $I-1$ degrees of freedom. Similarly, the column main effect, $n\Phi_J^2$ asymptotically follows a χ^2_{J-1} distribution while the tube main effect, $n\Phi_K^2$, follows a χ^2_{K-1} distribution. The bivariate terms of the partition - $n\Phi_{IJ}^2$, $n\Phi_{IK}^2$ and $n\Phi_{JK}^2$ - are measures of the row-column, row-tube and column-tube association, respectively, and are asymptotically chi-squared random variables with $(I-1)(J-1)$, $(I-1)(K-1)$ and $(J-1)(K-1)$ degrees of freedom, respectively. The last term of the partition, $n\Phi_{IJK}^2$ is the measure of three-way, or *trivariate*, association between all three variables and is asymptotically a chi-squared random variable with $(I-1)(J-1)(K-1)$ degrees of freedom. This general framework allows one to consider a mixture of tests, in particular goodness-of-fit tests, and association tests of the various relationships among the variables.

5 Example

To illustrate our partition briefly, we consider Pearson's index under *Scenario 0*; see Equation (4). The data concerns a study about the justice court delay in Italy [5]. We investigate the association among *Trial length*, *Subjects* of trials and *Number of Hearings*. The *Trial length* has four categories *low* duration (from 88 to 596 days), middle-low (from 597 to 1130 days), *mlow*, middle-long (from 1131 to 1950 days), *mlong*, and *long* (from 1951 to 6930 days) duration. The column variable, *Sub-*

ject of trials, has three categories *Obbligation, Controversy*, and *Real Rights* and also the tube variable has three categories, low number of hearings (from 0 to 2), *Hlow*, middle number of hearing (from 3 to 5), *Hmedium*, and high number of hearings (from 6 to 25), *Hhigh*. For a priori knowledge of the problem, we set the row probabilities as estimated from the data, the column probabilities as uniform, i.e. $p_{\bullet j \bullet} = (1/3, 1/3, 1/3)$, and the tube probabilities as equal to $p_{\bullet\bullet k} = (0.4, 0.4, 0.2)$. The overall association is $n\Phi^2 = \chi^2 = 4122$ with df=32 there exists strong evidence to say that it is statistically significant (p-value < 0.0001). The size and percentage contribution to χ^2 and the eight terms are reported in Table 2.

Table 1 Justice data: Crosstabulation of *Trial length* by *Subject* and by *Number Hearing*.

Hlow- *Low number of hearings*			
	Subject		
Trial Length	**Obbligation**	**Controversy**	**Real Right**
low	123	26	8
mlow	63	23	19
mlong	161	27	33
long	69	69	22
Hmedium- *Middle number of hearings*			
	Subject		
Trial Length	**Obbligation**	**Controversy**	**Real Right**
low	110	9	9
mlow	82	7	4
mlong	74	6	15
long	32	11	10
Hhigh- *High number of hearings*			
	Subject		
Trial Length	**Obbligation**	**Controversy**	**Real Right**
low	181	0	8
mlow	241	4	28
mlong	126	1	27
long	183	6	70

Table 2 Partition of the three-way chi-squared association measure under Scenario 0.

	X_J^2	X_K^2	X_{IJ}^2	X_{IK}^2	X_{JK}^2	X_{IJK}^2	X^2
Index	1591.15	869.84	92.94	152.93	1202.43	212.92	4122.22
% of Inertia	38.59	21.10	2.26	3.71	29.17	5.17	100.00
df	2.00	2.00	6.00	6.00	4.00	12.00	32.00
p-value	0.00	0.00	0.00	0.00	0.00	0.00	0.00

This output shows that the most dominant contributor (38.6%) to the association among the univariate terms of Equation (4) is due to the inclusion of the *Subject* variable. The unique not significant term is due to the row variable *Trial Length* (whose hypothesized probabilities are set equal to the estimated frequencies), all the others are statistically significant and the most important concerns the *Subject* of trials. Of the bivariate association terms from the partition, the most important is due to the column and tube variables, *Subject* and *Number of Hearing*, it is about 29% of the total association, while the least important bivariate association concerns the row and tube variables, *Trial Length* and *Subject*, that is 2.3% of the total association among the variables. Finally, the contribute of the trivariate association term is low (around 5%).

6 Discussion

This paper has proposed a general expression of the partition for three-way association statistics, in particular for the traditional three-way Pearson statistic (see Equation 4) when hypothesising complete independence model. Further investigation of the partitions when considering complete and partial independence will lead to other quite distinct situations in partitioning symmetric and asymmetric three-way association indices [14, 13]. Comparisons with other goodness-of-fit statistics for large-sample, like the likelihood-ratio statistic or the Wald statistic [11] will be pursued.

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