Cumulative chi-squared statistics for the service quality improvement: new properties and tools for the evaluation

Il chi quadrato cumulato per la qualità dei servizi: nuove proprietà e strumenti per la valutazione

Antonello D'Ambra, Antonio Lucadamo, Pietro Amenta, Luigi D'Ambra

Abstract In service quality evaluation, data are often categorical variables with ordered categories and collected in two way contingency table. The Taguchi's statistic is a measure of the association between these variables as a simple alternative to Pearson's test. An extension of this statistic for three way contingency tables handled in two way mode is introduced. We highlight its several properties, the approximated distribution, a decomposition according to orthogonal quantities reflecting the main effects and the interaction terms, and an extension of cumulative correspondence analysis based on it.
Abstract *Nella valutazione della qualità dei servizi erogati, i dati rappresentano spesso variabili qualitative ordinali raccolte in tabelle di contingenza a due vie. L'indice di Taguchi è una misura dell'associazione esistente tra queste variabili e nasce come un'alternativa al test di Pearson in presenza di variabili ordinali. In questo lavoro viene presentata una estensione di questo indice per tabelle di contingenza a tre vie. Se ne evidenziano diverse proprietà, la distribuzione approssimata, una decomposizione rispetto a quantità che riflettono gli effetti principali e l'interazione, nonchè un'estensione dell'analisi delle corrispondenze.*

Key words: Two and three way contingency tables, Chi-squared statistic, Taguchi's statistic, Interaction term, Main effects, Service quality evaluation.

Antonello D'Ambra
Department of Economics, Second University of Naples, Italy, Corso Gran Priorato di Malta, Capua, e-mail: antonello.dambra@unina2.it

Antonio Lucadamo, Pietro Amenta
Department of Law, Economics, Management and Quantitative Methods, University of Sannio, Italy, Piazza Arechi II, Benevento, e-mail: alucadam@unisannio.it, amenta@unisannio.it

Luigi D'Ambra
Department of Economics, Management and Institutions, University of Naples, Italy, Via Cinthia Monte Sant'Angelo, Napoli, e-mail: dambra@unina.it

1 Introduction

Service companies have given increasing importance to customer satisfaction (hereafter CS) over the years worldwide. Measuring the quality of a service is indeed a fundamental and strategic function for every firms because it allows checking the level of efficiency and effectiveness perceived by users. In service quality evaluation, data are often categorical variables with ordered categories and usually collected in two way contingency table. To determine the nature of the association, tests involving the Pearson chi-squared statistic are generally considered. However, the statistic does not take into account the structure of ordered categorical variables [1]. To overcome this problem, Taguchi [6, 7] developed a simple statistic that does take into consideration the structure of an ordered categorical variable. It does so by considering the cumulative frequency of the cells of the contingency table across the ordered variable. An extension of this statistic for three way contingency tables handled in two way mode is introduced in section 4, highlighting some properties and its approximated distribution. Moreover, an extension of correspondence analysis based on the suggested new statistic is proposed to study the association from a graphical point of view. It highlights the impacts of the main effects and the interaction terms on the association. This is obtained in section 5 by means of a decomposition of the new statistic according to orthogonal quantities reflecting several effects.

An application on real data about service quality evaluation using all the theoretical results will be shown in the extended version of this paper.

2 Notations

Let A, B, and Y be categorical variables with $i=1,\ldots,I$, $k=1,\ldots,K$ and $j=1\ldots,J$ categories, respectively, and suppose $(A_1,B_1,Y_1),\ldots,(A_n,B_n,Y_n)$ is a random sample of the random vector (A,B,Y). The basic data structures in this paper are two and three-way contingency tables $\mathbf{N}$ and $\check{\mathbf{N}}$ of orders (I,J) and (I,K,J) with frequencies $\{n_{ij}\}$ and $\{n_{ikj}\}$ counting the numbers of observations that fall into the cross-categories $i\times j$ and $i\times k\times j$, respectively. $\mathbf{N}$ cross classifies n statistical units according to two categorical variables A and Y while $\check{\mathbf{N}}$ according to three categorical variables A, B, and Y. Table $\check{\mathbf{N}}$ is handled in this paper in two way mode by row unfolding it according to variables A and B: the resulting two way contingency table $\tilde{\mathbf{N}}$ is then of size $[(I\times K)\times J]$ with general term n_{ikj}. We consider row variables A and B as predictors and the column variable Y as response, reflecting a unidirectional association between the categorical variables (rows versus column). Moreover, suppose that Y has an ordinal nature with increasing scores.

We denote by p_{ij} the probability of having an observation fall in the i-th row and j-th column of the table, with $\mathbf{P}=\{p_{ij}=n_{ij}/n\}$. $p_{i.}=\sum_{j=1}^{J}p_{ij}$ and $p_{.j}=\sum_{i=1}^{I}p_{ij}$ denote the probabilities that A and Y are in categories i and j, respectively.

Considering the two way table $\mathbf{N}$, let $z_{is}=\sum_{j=1}^{s}n_{ij}$ and $z_{.s}=\sum_{j=1}^{s}n_{.j}$ be the cumulative count and the cumulative column total up to the j-th column category, respectively, with $s = 1,\ldots,J-1$. $d_s = z_{.s}/n$ denotes the cumulative column proportion. Let $\tilde{\mathbf{P}} = \{p_{ikj} = n_{ikj}/n\}$ be the joint relative frequency distribution. Moreover, let $\mathbf{D}_I = \{p_{i.} = \sum_{j=1}^{J} p_{ij}\}$ and $\mathbf{D}_J = \{p_{.j} = \sum_{i=1}^{I} p_{ij}\}$ be diagonal matrices containing the row and column sum of $\tilde{\mathbf{P}}$, respectively. Let $\mathbf{D}_{IK} = diag(p_{ik.})$ (marginal row) and $\mathbf{D}_J = diag(p_{..J})$ (marginal column) be also the diagonal matrices with generic elements $p_{ik.} = \sum_{j=1}^{J} p_{ikj}$ and $p_{..j} = \sum_{i=1}^{I}\sum_{k=1}^{K} p_{ikj}$, respectively.

Lastly, denote $C_{iks} = \sum_{j=1}^{s} n_{ikj}$ with $s = 1,\ldots,J-1$ the cumulative frequencies of the $\{ik\}$-th row category up to the s-th column categories. Their consideration provides a way of ensuring that the ordinal structure of the column categories is preserved. Similarly, denote, $\tilde{d_s} = \sum_{j=1}^{s} p_{..j}$ the cumulative relative frequency up to the s-th column category.

3 The Taguchi's statistics in a nutshell

Taguchi [6, 7] proposed a measure of the association between categorical variables where one of them possesses ordered categories by considering the cumulative sum of cell frequencies across this variable. He introduced this measure as a simple alternative to Pearson's test in order to consider the impact of differences between adjacent ordered categories on the association between row and column categories. In order to assess the unidirectional association between the row and (ordered) column variables, Taguchi [6, 7] proposed the following statistic

$$T = \sum_{s=1}^{J-1} \frac{1}{d_s(1-d_s)} \sum_{i=1}^{I} n_{i.} \left(\frac{z_{is}}{n_{i.}} - d_s \right)^2 \quad (1)$$

with $0 \leq T \leq [n(J-1)]$. This statistic performs better than Pearson's chi-squared statistic when there is an order in the categories on the columns of the contingency table and it is more suitable for studies (such as clinical trials) where the number of categories within a variable is equal to (or larger than) 5 [8].

Takeuchi and Hirotsu [8] and Nair [3] showed also that the T statistic is linked to the Pearson chi-squared statistic $T = \sum_{s=1}^{J-1} \chi_s^2$ where χ_s^2 is Pearson's chi-squared for the $I \times 2$ contingency tables obtained by aggregating the first s column categories and the remaining categories $(s+1)$ to J, respectively. For this reason, the Taguchi's statistic T is called the *cumulative chi-squared statistic* (hereafter CCS). Nair [3] considers then the class of CCS-type statistics

$$T_{CCS} = \sum_{s=1}^{J-1} w_s \left[\sum_{i=1}^{I} n_{i.} \left(\frac{z_{is}}{n_{i.}} - d_s \right)^2 \right] \quad (2)$$

corresponding to a given set of weights $w_s > 0$. The choice of different weighting schemes defines the members of this class. Examples of possible choices for w_j are to assign constant weights to each term (i.e. $w_s = 1/J$) or assume it proportional to the inverse of the conditional expectation of the s-th term under the null hypothesis of independence (i.e. $w_s = [d_s(1-d_s)]^{-1}$). It is evident that T_{CCS} subsumes T in the latter case. Moreover, Nair shows that T_{CCS} with $w_s = 1/J$ (that is $T_N = \sum_{s=1}^{J-1}(1/J)\sum_{i=1}^{I} N_{i.}(z_{is}/n_{i.} - d_s)^2$) has good power against ordered alternatives.

Nair [3, 4] highlighted the main properties of the CCS-type tests by means of a matrix decomposition of this statistic into orthogonal components. Lastly, Taguchi's statistics can be also viewed as an approximate sum of likelihood ratios within the regression model for binary dependent variables following a scaled binomial distribution, providing in this way a different interpretation of this statistic [2]. Refer to [2] for a wider and deeper study with other new interpretations and characteristics of this statistic.

4 Cumulative Correspondence Analysis and Taguchi's Statistics for three way contigency tables handled in two way mode

In this paper we introduce a new extension of the Taguchi's statistic on a three-way contingency table, where one of the variables consists of ordered responses, handled in two way mode. We name "Multiple Taguchi's statistic" the following measure of the unidirectional association between the rows and (ordered) column variables

$$T^M = \sum_{s=1}^{J-1} \frac{1}{\tilde{d}_s(1-\tilde{d}_s)} \left[\sum_{i=1}^{I} \sum_{k=1}^{K} n_{ik.} \left(\frac{C_{iks}}{n_{ik.}} - \tilde{d}_s \right)^2 \right] \qquad 0 \le T_M \le n(J-1) \tag{3}$$

Likewise formulas (1) and (2) it is also possible to consider a class of CCS-type statistics T^M_{CSS} corresponding to a given set of weights $w_s > 0$. The choice of different weighting schemes defines the members of this class.

It is possible to show that there is a link between Multiple Taguchi's statistic, Pearson Chi-Squared statistic and C-Statistics $T^M = \sum_{s=1}^{J-1} \chi^2(s) = \frac{n}{n-1}\sum_{s=1}^{J-1} C(s)$. Here $\chi^2(s)$ and $C(s) = (n-1)[\sum_{i=1}^{I}\sum_{j=1}^{s} p_{ik.}\left(p_{ikj}/p_{ik.} - p_{..j}\right)^2]/(1-\sum_{j=1}^{J} p_{..j}^2)$ are the Pearson chi-squared and the C-statistics, respectively, for a $[(I \times K) \times 2]$ contingency table obtained by aggregating the first j column categories and the remaining categories $(j+1)$ to J.

It is possible to highlight other properties by means of a matrix decomposition of the CCS-type statistic T^M_{CSS} into orthogonal components. For instance, this allows to introduce the Multiple Taguchi's statistic at heart of a new cumulative extension of correspondence analysis (hereafter MTA). Main goal of MTA is to show how similar cumulative categories are with respect to joined nominal ones from a graphical point of view. We represent the variations of column categories rather than the categories on the space generated by cumulative frequencies. Let define the matrix

$$\mathbf{R} = \mathbf{D}_{IK}^{-1}\tilde{\mathbf{P}}\mathbf{A}^T\mathbf{W}^{\frac{1}{2}} \tag{4}$$

where $\mathbf{W}$ is a diagonal square matrix of dimension $[(J-1)\times(J-1)]$ with general term w_s and $\mathbf{A}$ the following $[(J-1)\times J]$ matrix

$$\mathbf{A} = \begin{bmatrix} 1-\tilde{d}_1 & -\tilde{d}_1 & \dots & -\tilde{d}_1 \\ 1-\tilde{d}_2 & 1-\tilde{d}_2 & \dots & -\tilde{d}_2 \\ \vdots & \vdots & \ddots & \vdots \\ 1-\tilde{d}_{J-1} & 1-\tilde{d}_{J-1} & \dots & -\tilde{d}_{J-1} \end{bmatrix}$$

$\mathbf{A}$ can be also written as $\mathbf{A} = \mathbf{M} - [\mathbf{D}(\mathbf{1}_{J-1}\mathbf{1}_J^T)]$ where $\mathbf{M}$ is unitriangular lower matrix of dimension $(J-1)\times J$, $\mathbf{D} = diag(\tilde{d}_s)$, with $\mathbf{1}_{J-1}$ and $\mathbf{1}_J$ column vectors of one of dimension $(J-1)$ and J, respectively. The CSS-type Multiple Taguchi's statistic T_{CSS}^M is then given by

$$T_{CSS}^M = n\times ||\mathbf{R}||^2_{\mathbf{D}_{IK}} = n\times ||\mathbf{D}_{IK}^{-1}\tilde{\mathbf{P}}\mathbf{A}^T\mathbf{W}^{\frac{1}{2}}||^2_{\mathbf{D}_{IK}} = n\times trace\left(\mathbf{D}_{IK}^{-\frac{1}{2}}\tilde{\mathbf{P}}\mathbf{A}^T\mathbf{W}\mathbf{A}\tilde{\mathbf{P}}^T\mathbf{D}_{IK}^{-\frac{1}{2}}\right)$$

Let $GSVD(\mathbf{R})_{\mathbf{D}_{IK},\mathbf{I}}$ denotes the generalized singular value decomposition of matrix $\mathbf{R} = \{r_{ikj}\}$ of rank M such that $\mathbf{R} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{V}^T$, where $\mathbf{U}$ is an $[I\times M]$ matrix of left singular vectors such that $\mathbf{U}^T\mathbf{D}_I\mathbf{U} = \mathbf{I}_M$, $\mathbf{V}$ is an $[(J-1)\times M]$ matrix of right singular vectors such that $\mathbf{U}^T\mathbf{U} = \mathbf{I}_M$ and $\boldsymbol{\Lambda}$ is a positive definite diagonal matrix of order M of singular values of $\mathbf{R}$ of general term λ_m $(m = 1,\dots,M)$. Total inertia is given by

$$||\mathbf{R}||^2_{D_{IK}} = trace(\mathbf{R}^T\mathbf{D}_{IK}\mathbf{R}) = \sum_{i=1}^{I}\sum_{k=1}^{K}\sum_{j=1}^{J} p_{ik.}r^2_{ikj} = \sum_{m=1}^{M}\lambda_m^2 = \frac{T_{CSS}^M}{n}$$

Finally, row and column standard coordinates for the graphical representation of the association between predictors and response categorical variables are then given by $\mathbf{F} = \mathbf{U}\boldsymbol{\Lambda}$ and $\mathbf{G} = \mathbf{V}\boldsymbol{\Lambda}$, respectively.

According to the Nair's approach [3, 4] we show how the distribution of T_{CSS}^M is approximated using Satterthwaite's method [5]. Let $\boldsymbol{\Gamma}$ be the $(J-1)\times(J-1)$ diagonal matrix of the nonzero singular-values of $\mathbf{A}^T\mathbf{W}\mathbf{A}$ and consider the singular value decomposition $\mathbf{A}^T\mathbf{W}^{\frac{1}{2}} = \mathbf{Q}\boldsymbol{\Gamma}\mathbf{Z}^T$ with $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ and $\mathbf{Z}^T\mathbf{Z} = \mathbf{I}$ such that $\mathbf{A}^T\mathbf{W}\mathbf{A} = \mathbf{Q}\boldsymbol{\Gamma}^2\mathbf{Q}^T$. The CSS-type Multiple Taguchi's statistic T_{CSS}^M is given by

$$\begin{aligned} T_{CSS}^M &= n\times trace\left(\mathbf{D}_{IK}^{-\frac{1}{2}}\tilde{\mathbf{P}}\mathbf{A}^T\mathbf{W}\mathbf{A}\tilde{\mathbf{P}}^T\mathbf{D}_{IK}^{-\frac{1}{2}}\right) = n\times trace\left(\mathbf{D}_{IK}^{-\frac{1}{2}}\tilde{\mathbf{P}}\mathbf{Q}\boldsymbol{\Gamma}^2\mathbf{Q}^T\tilde{\mathbf{P}}^T\mathbf{D}_{IK}^{-\frac{1}{2}}\right) \\ &= n\times trace\left(\mathbf{S}\boldsymbol{\Gamma}^2\mathbf{S}^T\right) \end{aligned}$$

where $\mathbf{S} = \mathbf{D}_I^{-\frac{1}{2}}\tilde{\mathbf{P}}\mathbf{Q}$. It is possible to show that the i-th elements S_{is} of column vector $\mathbf{S}_s$ are asymptotically iid with a $N(0,1)$ distribution as $n\to\infty$, with $s = 1,\dots,J-1$ and $i = 1,\dots,(I\times K)-1$. Then, under the hypothesis of homogeneity and given the row and column probabilities, the components $\mathbf{S}_s^T\mathbf{S}_s = \sum_{i=1}^{I\times K} S_{is}^2$ are asymptotically iid with a $\chi^2_{[(I\times K)-1]}$ distribution. Consequently, under the null hypothesis, the limiting distribution of the CSS-type Multiple Taguchi's statistic T_{CSS}^M is a linear combination of chi-squared distributions

$$T^{M}_{CSS} = n \times trace\left(\mathbf{S}\boldsymbol{\Gamma}^2\mathbf{S}^T\right) \xrightarrow[H_0]{d} \sum_{s=1}^{J-1} \gamma_s \times \chi^2_{[(I\times K)-1]}(s)$$

where $\chi^2_{[(I\times K)-1]}(s)$ is the chi-squared distribution for the s-th component $\mathbf{S}_s^T\mathbf{S}_s$ $(s = 1,\ldots,J-1)$ and γ_s are elements of matrix $\boldsymbol{\Gamma}$. By using Satterthwaite's two-moment approximation [5], the asymptotic distribution of T^M_{CSS} can be then approximated [3, 4] by $d^M_{CSS} \times \chi^2_{v^M_{CSS}}$ with $v^M_{CSS} = (d^M_{CSS})^{-1}\sum_{s=1}^{J-1}\gamma_s$ degrees of freedom and $d^M_{CSS} = [(I\times K)-1]^{-1}(\sum_{s=1}^{J-1}\gamma_s^2/\sum_{k=1}^{J-1}\gamma_s)$.

5 Orthogonal decomposition of Multiple Taguchi's statistic

The Multiple Taguchi's statistic is a measure of association that contains both main effects and interaction term. The main effects represent the change in the response variables due to the change in the level/categories of the predictor variables, considering the effects of their addition. The interaction effect represents the combined effect of predictor categorical variables on the ordinal response variable.

The interpretation of MTA graphical results can be improved if we highlight the impact of these effects on the association. The Multiple Taguchi's statistic can be then decomposed in different orthogonal quantities:

$$T^M_{CSS} = T^{A\cup B} + T^{A\times B} = T^A + T^{B|A} + T^{A\times B} = T^{A|B} + T^B + T^{A\times B}$$

where $T^{A\cup B}$ reflects the main effects and represents the change in the response variables due to the change on the levels/categories of the predictor variables considering their joining effects, T^A (or T^B) represents the Taguchi's statistic calculated between Y and A (or B) after a row aggregation of variable B (or A), while $T^{B|A}$ (or $T^{A|B}$) is the Taguchi's statistic between Y and B (or A) where the effects of variable A has been partialled out (or B). Finally, $T^{A\times B}$ is the interaction effect and represents the combined effect of predictor variables on the response variable. In particular, there is an interaction between two predictor variables when the effect of one predictor variable varies as the levels/categories of the other predictor vary. If the interaction is not significant, it is possible to examine the main effects. Instead, if the interaction is statistically significant and of strong entity, then, it is not useful to consider the main effects.

In order to separate the main effects and the interaction term, the approach starts from a constraints matrix. Let $\mathbf{T}_{A\cup B} = [\mathbf{T}_A|\mathbf{T}_B]$ be the matrix of dummy variables with $\mathbf{T}_A = (\mathbf{1}_K \otimes \mathbf{I}_I)$ (factor A), $\mathbf{T}_B = (\mathbf{I}_K \otimes \mathbf{1}_I)$ (factor B) and such that formula (4) can be written as $\mathbf{R} = \mathbf{D}_{IK}^{-1}\mathbf{H}^T_{1/D_{IK}}\tilde{\mathbf{P}}\mathbf{M}^T\mathbf{W}^{\frac{1}{2}}$ where $\mathbf{H}_{1/D_{IK}} = \mathbf{I}_{IK} - [\mathbf{1}_{IK}(\mathbf{1}^T_{IK}\mathbf{D}_{IK}\mathbf{1}_{IK})^{-1}\mathbf{1}^T_{IK}\mathbf{D}_{IK}]$ is the orthogonal projector onto the null space of $\mathbf{1}_{IK}$ in metric $\mathbf{D}_{IK}$ with $\mathbf{1}_{IK}$ unitary column vectors of dimension IK. $\mathbf{H}_{1/D_{IK}}$ eliminates the row marginal effect from the relationship between rows and columns. The main effects are given by

$$\mathbf{R}_{A\cup B} = \mathbf{H}_{1/D_{IK}}\mathbf{T}_{A\cup B}(\mathbf{T}_{A\cup B}^T\mathbf{H}_{1/D_{IK}}^T\mathbf{D}_{IK}\mathbf{T}_{A\cup B})^{-1}\mathbf{T}_{A\cup B}^T\mathbf{H}_{1/D_{IK}}^T\tilde{\mathbf{P}}\mathbf{M}^T\mathbf{W}^{\frac{1}{2}}$$

Since $\mathbf{R}_{A\times B} = \mathbf{R} - \mathbf{R}_{A\cup B}$ then we obtain the following norm decomposition

$$||\mathbf{R}||_{D_{IK}}^2 = ||\mathbf{R}_{A\cup B}||_{D_{IK}}^2 + ||\mathbf{R}_{A\times B}||_{D_{IK}}^2 \quad (5)$$

Similarly, a double decomposition of the main effects in orthogonal quantities is

$$||\mathbf{R}_{A\cup B}||_{D_{IK}}^2 = ||\mathbf{R}_A||_{D_{IK}}^2 + ||\mathbf{R}_{B|A}||_{D_{IK}}^2 = ||\mathbf{R}_B||_{D_{IK}}^2 + ||\mathbf{R}_{A|B}||_{D_{IK}}^2 \quad (6)$$

where $\mathbf{R}_A = \mathbf{H}_{1/D_{IK}}\mathbf{T}_A(\mathbf{T}_A^T\mathbf{H}_{1/D_{IK}}^T\mathbf{D}_{IK}\mathbf{T}_A)^{-1}\mathbf{T}_A^T\mathbf{H}_{1/D_{IK}}^T\mathbf{P}\mathbf{M}^T\mathbf{W}^{\frac{1}{2}}$ and $\mathbf{R}_B = \mathbf{H}_{1/D_{IK}}\mathbf{T}_B$ $(\mathbf{T}_B^T\mathbf{H}_{1/D_{IK}}^T\mathbf{D}_{IK}\mathbf{T}_B)^{-1}\mathbf{T}_B^T\mathbf{H}_{1/D_{IK}}^T\mathbf{P}\mathbf{M}^T\mathbf{W}^{\frac{1}{2}}$. Decomposition (6) shows that $A\cup B \neq (A+B)$ because $A\cup B = (A+B|A) = (B+A|B)$ since A and B are not orthogonal factors [9]. If we consider a balanced design then we have $R_{A|B} = R_A$ and $R_{B|A} = R_B$ so that we can write $||\mathbf{R}_{A\cup B}||_{D_{IK}}^2 = ||\mathbf{R}_A||_{D_{IK}}^2 + ||\mathbf{R}_B||_{D_{IK}}^2$ and decomposition (5) is now $||\mathbf{R}||_{D_{IK}}^2 = ||\mathbf{R}_A||_{D_{IK}}^2 + ||\mathbf{R}_B||_{D_{IK}}^2 + ||\mathbf{R}_{A\times B}||_{D_{IK}}$.

Table 1 Multiple Taguchi's statistic decomposition

Decomposition	Index	$\tilde{d}$	Statistic	degrees of freedom
Main effects	$T^{A\cup B}$	$\tilde{d}^{A\cup B} = \left(\frac{1}{\tilde{d}^A} + \frac{1}{\tilde{d}^{B\|A}}\right)^{-1}$	$T^{A\cup B}/\tilde{d}^{A\cup B}$	$v^{A\cup B} = \frac{1}{\tilde{d}^{A\cup B}}\sum_{s=1}^{J-1}\gamma_s$
Interaction	$T^{A\times B}$	$\tilde{d}^{A\times B} = \left(\frac{1}{\tilde{d}^M} - \frac{1}{\tilde{d}^A} - \frac{1}{\tilde{d}^{B\|A}}\right)^{-1}$	$T^{A\times B}/\tilde{d}^{A\times B}$	$v^{A\times B} = \frac{1}{\tilde{d}^{A\times B}}\sum_{s=1}^{J-1}\gamma_s$
Total	T_{CSS}^M	$\tilde{d}_{CSS}^M = \frac{1}{[(I\times K)-1]}\frac{\sum_{s=1}^{J-1}\gamma_s^2}{\sum_{s=1}^{J-1}\gamma_s}$	$T_{CSS}^M/\tilde{d}_{CSS}^M$	$v_{CSS}^M = \frac{1}{\tilde{d}_{CSS}^M}\sum_{s=1}^{J-1}\gamma_s$

Table 2 Alternative $T^{A\cup B}$ decompositions

Decomposition	Index	$\tilde{d}$	Statistic	degrees of freedom
Factor A	T^A	$\tilde{d}^A = \frac{1}{(I-1)}\frac{\sum_{s=1}^{J-1}\gamma_s^2}{\sum_{s=1}^{J-1}\gamma_s}$	$T^A/\tilde{d}^A$	$v^A = \frac{1}{\tilde{d}^A}\sum_{s=1}^{J-1}\gamma_s$
Factor A|B	$T^{A\|B}$	$\tilde{d}^{A\|B} = \frac{1}{(I-1)}\frac{\sum_{s=1}^{J-1}\gamma_s^2}{\sum_{s=1}^{J-1}\gamma_s}$	$T^{A\|B}/\tilde{d}^{A\|B}$	$v^{A\|B} = \frac{1}{\tilde{d}^{A\|B}}\sum_{s=1}^{J-1}\gamma_s$
Factor B	T^B	$\tilde{d}^B = \frac{1}{(K-1)}\frac{\sum_{s=1}^{J-1}\gamma_s^2}{\sum_{s=1}^{J-1}\gamma_s}$	$T^B/\tilde{d}^B$	$v^B = \frac{1}{\tilde{d}^B}\sum_{s=1}^{J-1}\gamma_s$
Factor B|A	$T^{B\|A}$	$\tilde{d}^{B\|A} = \frac{1}{(K-1)}\frac{\sum_{s=1}^{J-1}\gamma_s^2}{\sum_{s=1}^{J-1}\gamma_s}$	$T^{B\|A}/\tilde{d}^{B\|A}$	$v^{B\|A} = \frac{1}{\tilde{d}^{B\|A}}\sum_{s=1}^{J-1}\gamma_s$

Let $\mathbf{R}_{(A\cup B)} = \mathbf{U}_{A\cup B}\boldsymbol{\Lambda}_{A\cup B}\mathbf{V}_{A\cup B}^T$ denote $GSVD(\mathbf{R}_{A\cup B})_{D_{IK},I}$ where $\mathbf{U}_{A\cup B}$ is an $(I\times M)$ matrix of right singular vectors such that $\mathbf{U}_{A\cup B}^T\mathbf{D}_{IK}\mathbf{U}_{A\cup B} = \mathbf{I}_M$ $[M = rank(\mathbf{R}_{A\cup B})]$,

$\mathbf{V}_{A\cup B}$ is $[(J-1)\times M]$ matrix of right singular vectors such that $\mathbf{V}^T_{A\cup B}\mathbf{V}_{A\cup B} = \mathbf{I}_M$ and $\boldsymbol{\Lambda}$ is a positive definite diagonal matrix of order M of singular value of $\mathbf{R}_{A\cup B}$ with general term $\lambda_m^{A\cup B}$ with $m = 1,\dots,M$. Row and column standard coordinates of the main effects are given by $\mathbf{F}_{A\cup B} = \mathbf{U}_{A\cup B}\boldsymbol{\Lambda}_{A\cup B}$ and $\mathbf{G}_{A\cup B} = \mathbf{V}_{A\cup B}\boldsymbol{\Lambda}_{A\cup B}$, respectively. It's also possible to plot the interaction term and the single effects in the same way.

6 Conclusion

Taguchi introduced his statistic as simple alternative to Pearson's chi-squared test for two way contingency tables. Actually, χ^2 does not perform well when we have a contingency table cross-classifying at least one ordinal categorical variable. In this paper an extension of this statistic for three way contingency tables handled in two way mode has been introduced highlighting some properties. The approximated distribution of the CCS-type Multiple Taguchi's statistic T^M_{CSS}, by using Satterthwaite's method, has been also suggested. In this paper, an extension of Correspondence Analysis based on the decomposition of the CCS-type Multiple Taguchi's statistic has been moreover proposed. The interpretation of the graphical results has been improved highlighting the impact of the main effects and the interaction terms on the association. This is obtained with a decomposition of statistic T^M_{CSS} according to orthogonal quantities reflecting several effects.
Finally, an extended version of this paper will include an application on real data about service quality evaluation. All the theoretical results will be used showing also the graphical outputs. We will be also able to evaluate the impact of the main effects and interaction term on association among the categorical variables.

References

1. Agresti A.: Categorical Data Analysis, third edition. John Wiley & Sons, (2013)
2. D'Ambra, L., Amenta, P., D'Ambra, A.: Decomposition of cumulative chi-squared statistics, with some new tools for their interpretation, Statistical Methods and Applications, doi 10.1007/s10260-017-0401-3, (2017)
3. Nair, V.N.: Testing in industrial experiments with ordered categorical data. Technometrics **28**(4), 283–291 (1986)
4. Nair, V.N.: Chi-squared type tests for ordered alternatives in contingency tables. Journal of American Statistical Association **82**, 283–291 (1987)
5. Satterthwaite, F.: An approximate distribution of estimates of variance components. Biometrical Bullettin (2), 110–114 (1946)
6. Taguchi, G.: Statistical Analysis. Tokyo: Maruzen (1966)
7. Taguchi, G.: A new statistical analysis for clinical data, the accumulating analysis, in contrast with the chi-square test. Saishin Igaku **29**, 806–813 (1974)
8. Takeuchi, K., Hirotsu, C.: The cumulative chi square method against ordered alternative in two-way contingency tables. Tech. Rep. 29, Reports of Statistical Application Research. Japanese Union of Scientists and Engineers (1982)
9. Takeuchi, K., Yanai, H., Mukherjee, B.N.: The Foundations of Multivariate Analysis. John Wiley & Sons (Asia) Pte Ltd., (1982)