Stochastic Dominance for Generalized Parametric Families

*Dominanza Stocastica per Famiglie Parametriche Generalizzate*

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**Abstract** The T-X family is a recent method for generating distributions by composing probability distributions and quantile functions. Such an approach makes it possible to obtain a large number of flexible families of parametric distributions, new or already existing, most of which are typically used to model phenomena in different areas, such as economics and finance. We present a general method to derive sufficient conditions for the second-order stochastic dominance, within T-X families of distributions.

**Abstract** *La famiglia T-X permette di generare distribuzioni di probabilità tramite la composizione di funzioni di ripartizione e funzioni quantile. Tale approccio permette di ottenere un gran numero di famiglie parametriche molto flessibili, nuove o già esistenti, gran parte delle quali sono utilizzate per descrivere fenomeni di tipo economico, finanziario, e non solo. Si presenta un metodo generale che permette di ricavare condizioni sufficienti per la dominanza del secondo ordine all’interno di famiglie T-X di distribuzioni.*

**Key words:** stochastic dominance, T-X family, generalized distributions

1. Introduction

Generalized parametric distributions have a wide range of application in several different scientific fields, due to the flexibility offered by a quite high number of parameters. Ranking generalized distribution with respect to a dominance relation represents a major issue in different areas, such as economics and finance (see e.g. Wilfling 1996, Kleiber and Kotz, 2003). On the other hand, it is generally difficult to rank generalized models with stochastic dominance rules, because their functional forms are generally not easily tractable.

In this framework, the so-called T-X family, recently introduced by Alzatreeh et al. (2013), provides an interesting method for generating distributions by composing a “baseline” distribution with a quantile function and finally a “transformer” distribution (see Aljarrah et al. 2014). This approach can be used to generate new or already existing families of distributions. It can be shown (Lando and Bertoli-Barsotti) that, by studying the composition of the T-X family, it is possible to derive sufficient conditions for the first order stochastic dominance (FSD) and the second order stochastic dominance (SSD). In particular, under some circumstances, a pair of T-X models still preserve the SSD order if the corresponding baseline and transformer distributions are ranked by FSD and SSD, respectively.

1. Preliminaries

In this paper, we refer only to continuous random variables (RVs). We denote with CDF the cumulative distribution function and with PDF the probability density function. Therefore, a RV has CDF and PDF . We recall the basic definitions of the first order stochastic dominance (FSD) and the SSD.

**Definition 1.** We say that FSD dominates and write iff

**Definition 2.** We say that SSD dominates and write iff

It is clear that FSD holds iff CDFs do not cross. Moreover, SSD can be related to the number of crossings between CDFs or PDFs. In particular, when the integral conditions of Def. 2 is difficult to verify for some parametric distributions, we may use an alternative method for deriving sufficient conditions for SSD, which requires CDFs to cross (at most) once (see Hanoch and Lèvi, 1969). Moreover, when it is not possible to verify the crossing condition on CDFs (e.g. when a closed form expression for the CDF is not available), we can rely on some closely related results, which involve densities. In particular, it is sufficient to prove that PDFs cross at most twice (Shaked 1982; see also Ramos et al. 2000, for some related conditions for non-negative RVs). Let us denote with the number of sign changes of a function . We summarize some important results in the theorem below.

**Theorem 1.**

Let and have finite means.

1. If and the sign sequence is , then iff .
2. Let with sign sequence Then, iff .
3. Let be non-negative RVs and let be unimodal, where the mode is a supremum. Then, implies .

The T-X{Y} method, originally introduced by Alzaatreh et al. (2013) and then studied by also Aljarrah et al. (2014), is based on the composition of the CDFs of two RVs, and , with the quantile function (QF) of a third RV . Given three RVs , and , where and must have the same support, a new RV is defined by means of its CDF

, (1)

where is the QF of . The corresponding PDF is

, (2)

where are the PDFs of , respectively (Aljarrah et al. 2014). In this formula, plays the role of the generator distribution (transformer) and represents a baseline distribution (transformed).

Many continuous RVs have closed-form expressions for the QF, than can be used as the RV in (1), to generate T-X{Y} families. For instance, the original paper of Alzaatreh et al. (2013) focuses on the T-X{exponential} family, which is obtained by taking to be the QF of an exponential RV with scale parameter equal to 1, i.e.:

. (3)

The T-X{Y} family makes it possible to generate a large number of new families of distributions, as well as many existing parametric models of noticeable practical relevance because of their several applications, such as: the generalized beta of the first and the second kind, and the generalized gamma distributions (McDonald 1984).

1. Sufficient conditions for SSD

Because many existing parametric distributions belong to the T-X family, we are concerned with finding the sufficient conditions for ranking distributions of such family with FSD and SSD (in particular). It can be shown (Lando ans Bertoli-Barsotti) T-X families obtained by composition of CDFs and QFs ranked by FSD or SSD preserve some kind of order.

In this study, we are interested in studying dominance relations among pairs of distributions within the same T-X family. Put otherwise, we compare pairs of distributions with CDF given by (1), but with different parameters. In particular, we assume that and , taken individually, are parametric families of distributions, say,

,, (3)

Thus, the new distribution defined by (1) depends on the parameters of and :

. (4)

We aim at comparing the RVs and , where, for :

, (5)

with and .

**Theorem 3.** Let , for . Let and have finite means, and let be convex . If , and with , then .

*Proof.* This theorem has been proved by Lando and Bertoli-Barsotti (submitted manuscript).

It can be shown that there is a wide class of distributions that can be ranked using the sufficient condition of Theorem 2, although, in some cases, the proposed method is not applicable, because, for some T-X families, is not convex.

Theorem 2 establishes that a strong dominance (FSD) between baseline distributions and a weak dominance (SSD) between generators may be sufficient for the weak dominance among the T-X family. Now, it is also worth noting that, for convex, , with single-crossing CDFs, and do not imply . This can be shown with a counter-example. A fortiori, and do not imply as well. Therefore, if we wish to rank by SSD with this method, it is generally required that the baseline distributions are ranked by FSD.

1. Conclusions

This study is aimed at deriving sufficient conditions for stochastic dominance within the T-X family of distributions. This approach can be extended to other types of dominance. In particular, we shall analyse the interesting case of the Lorenz order, which is especially relevant in the field of economics.

**Funding**

The research was supported through the Czech Science Foundation (GACR) under project 17-23411Y (to T.L.).

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